UNIVERSITI SAINS MALAYSIA

**BAHAGIAN PENYELIDIKAN DAN INOVASI**
RESEARCH AND INNOVATION DIVISION



*Ruj. Kami*: FPP 2005/ 187 [G0003]
*Tarikh* : 28 Julai 2008

Prof. Madya Rosy Teh Chooi Gim
Pusat Pengajian Sains Fizik
Universiti Sains Malaysia

Tuan/Puan

**Laporan Akhir Geran Scientific Advancement Grant Allocation ( SAGA)**
**Tajuk Projek : " A Study of the Effects Of Gravity Upon The Yang Mills – Higgs Systems"**
**No. Akaun : 304/PFIZIK/653004/A118**

Dengan segala hormatnya perkara di atas dirujuk.

Terlebih dahulu suka saya ucapkan terima kasih di atas laporan akhir geran Scientific Advancement Grant Allocation (SAGA) yang disampaikan seperti tajuk di atas.

Seterusnya walaupun projek ini telah selesai, Jabatan Bendahari telah dinasihatkan untuk menangguhkan penutupan akaun projek kepada **31 Ogos 2008**. Tempoh ini diberi untuk membolehkan penjelasan semua urusan tuntutan dan bayaran yang telah dikomitkan di dalam tempoh projek. Walaubagaimanapun, tuan/puan dinasihatkan supaya tidak mengeluarkan borang-borang pesanan baru di dalam tempoh ini.

Selanjutnya sila ambil perhatian terhadap perkara-perkara berikut sekiranya berkaitan :

(i) semua penerbitan harus merakamkan penghargaan kepada geran penyelidikan Scientific Advancement Grant Allocation (SAGA) dan tuan/puan dipohon mengemukakan satu salinan ke pejabat RCMO; dan

(ii) pihak kami akan mengagihkan semula peralatan yang telah dibeli menggunakan peruntukan geran ini seandainya terdapat penyelidik lain yang memerlukan peralatan tersebut.

Harap maklum, projek ini dianggap telah selesai dengan jayanya.

Sekian, terima kasih.

**"BERKHIDMAT UNTUK NEGARA"**
*"Bersaing Di Peringkat Dunia: Komitmen Kita"*

CHE MERAH ISMAIL
Penolong Pegawai Tadbir Kanan

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AIY 2/3/09

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Pejabat Pelantar Penyelidikan

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Pusat Pengajian Sains Fizik

Prof. Madya Mohd. Nawawi Mohd Nordin
**Timbalan Dekan** (Pengajian Siswazah & Penyelidikan)
Pusat Pengajian Sains Fizik

Cik Nursyatina Abdul Raof
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Pejabat Pelantar Penyelidikan

Encik Mohd Pisol Ghadzali
**Pemangku Ketua Pustakawan**
Perpustakaan Hamzah Sendut 1

Disampaikan satu salinan laporan akhir projek untuk simpanan Perpustakaan

Puan Ansuya a/p Narhari
**Penolong Bendahari**
Unit Kumpulan Wang Penyelidikan
Jabatan Bendahari

Sila ambil tindakan menutup akaun projek pada **31 Ogos 2008** dan sila kemukakan satu salinan kewangan terakhir ke pejabat (RCMO)

*CMI/liza...laporanakhirsaga/08*

Round 1

AKADEMI SAINS MALAYSIA

# SAGA END OF PROJECT REPORT – To be filled by principal Investigator

**Date of this report: July 4, 2007**

## A. DETAILS OF THE SAGA PROJECT


Principal Investigator : **Assoc Prof Dr Rosy Teh Chooi Gim***
**Pusat Pengajian Sains Fizik**
**Universiti Sains Malaysia**
**11800 USM**
**Pulau Pinang**
**Tel: 04-653 3678**
**Fax: 04-6579 150**
**Email: rosyteh@usm.my**




Co-researcher : i) **Wong Khai Ming (USM)**
*(**Please state names of additional co-researchers involved since project started [if any])* ii) ....................................

SAGA Fund Reference No. : **P 43**

Title : **A study of the effect of gravity upon the Yang-Mills-Higgs Systems**

Grant Approved : **RM 125,587.55**

1st Disbursement : **RM 79,948.60**

2nd Disbursement : **RM 45,638.95**

Project Duration : 24 months. Start Date: **1 April 2005** End Date: **1 April 2007**

This project was completed : a ■ within the period originally proposed; or

b ☐ extended beyond the proposed period

How many man-months did the project involve?

| 2 | 4 | Man-months

What were the total project expenses? **RM 122,911.87**

## B. OBJECTIVES ACHIEVEMENT

i) Original project objectives (please state the specific project objectives as described in the application form)

1. To study the relationship between the numerical MAP solutions of Burkhard Kleihaus and Jutta Kunz and the exact analytic monopole-antimonopole solutions which we have found with our ansatz in the limit of vanishing Higgs potential.

2. To study the effect of gravity on these monopole-antimonopole solutions.

Basic Research Project

- [x] Algorithm
- [x] Structure
- [ ] Data
- [ ] Other, please specify : ______________________________

b. How would you characterize the quality of this output?

- [ ] Significant breakthrough
- [x] Major Improvement
- [ ] Minor Improvement

ii. Contribution of the project to knowledge

a. How has the output of the project been documented?

- [ ] Detailed project report
- [ ] Product/process specification documents
- [x] Other, please specify: A compilation of papers from which we will write out a book in the very near future.

ii) How significant are citations of the results?

- [ ] Citations in national publications
- [ ] Citations in international publications
- [x] None yet
- [ ] Not known

## H. REPORTS, PAPERS AND PUBLICATIONS

### i. List of reports and conference/seminar papers written:

**1.** Rosy Teh and K.M. Wong, "*A-M-A Monopole Configurations*", Contributed paper for the International Meeting on Frontiers of Physics (IMFP), 25-29 July 2005, Mines Beach Resort and Spa, Kuala Lumpur.

**2.** Rosy Teh and K.M. Wong, "*Half-Integer Topological Charge Multimonopole*", Contributed paper for the International Meeting on Frontiers of Physics (IMFP), 25-29 July 2005, Mines Beach Resort and Spa, Kuala Lumpur.

**3.** Rosy Teh, "*A Note on the Julia-Zee Dyon Solution*", Proceedings of the IV. International Symposium, QUANTUM THEORY AND SYMMETRIES, (QTS - 4) 15 - 21 August 2005, Varna, Bulgaria; KEK Preprint: TH-1016; hep-th/0505058

**4.** Rosy Teh and K.M. Wong, "*Dyons of One Half Monopole Charge*, hep-th/0506047; Contributed paper for the International Advanced Technology Congress 2005, (CATS 2005), IOI Marriott Hotel, Putrajaya, 6-8 December 2005.

**5.** K.M. Wong and Rosy Teh, *A Numerical Investigation of the MAP Solution of the SU(2) YMH Theory*, paper for the National Conference of Physics (PERFIK 2006), Dec 6-7, 2006, Palace of the Golden Horses, Kuala Lumpur, Malaysia.

**6.** Rosy Teh and K.M. Wong, *Some Comments on the Monopole, MAP and MAC Solutions of the YMH Theory*, paper for the National Conference of Physics (PERFIK 2006), Dec 6-7, 2006, Palace of the Golden Horses, Kuala Lumpur, Malaysia.

NOTE: A copy of the six papers is attached.

**ii. List of scientific publications** (including name(s) of co-author(s), date of publication, location and name of publisher. Please attach pre-print copies of the publications)

**1.** Rosy Teh and K.M. Wong, "*A-M-A Monopole Configurations*", Jurnal Fizik Malaysia Vol. 27, No. 3 (2006).

**2.** Rosy Teh and K.M. Wong, "*Dyons of One Half Monopole Charge*, hep-th/0506047; Int. J. Mod. Phys A, Vol. 21, No. 26, 5285-5298, (2006)

**3.** Rosy Teh and K.M. Wong, "Monopole-antimonopole and their Vortex Rings"; J. Math. Phys. **46**, 082301 (August 2005) (*13 pages*)

NOTE: A copy of the three papers is attached.

## C. FINANCIAL REPORT

**i) Please report the expenditure of the project so far:** (Penyata Perbelanjaan 31 May 2007)

| Vote | Project Cost Component | Expenditure RM Allocated | Expenditure RM Expended |
|---|---|---|---|
| Vote 11000 | Salary and wage | 72,482.00 | 53,098.79 |
| Vote 52000 | Vote 52000 | 0 | 316.98 |
| Vote 21000 | Traveling expenses and subsistence | 15,000.00 | 6,020.20 |
| Vote 22000 | Transportation of goods | NIL | NIL |
| Vote 23000 | Communication and utilities i.e. Phone, fax, postage etc | NIL | NIL |
| Vote 24000 | Rental | NIL | NIL |
| Vote 26000 | Supply of raw materials& materials for repair & maintenance | 4,000.00 | 0 |
| Vote 27000 | Research materials & supplies including animals, disposables etc. | 0 | 471.00 |
| Vote 28000 | Maintenance and minor repair services | 1,000.00 | 0 |
| Vote 29000 | Professional services & other services including printing & hospitality, registration fees | 13,105.00 | 11,065.00 |
| Vote 35000 | Equipment (Justification and quotation of each equipment to be purchased should be provided) | 20,000.00 | 41,673.00 |
| **Total** | | 125,587.00 | 112,644.97 |

- *Please attach financial statement/ breakdown from bursary of institution.*
- *If there is amount transferred to another institution for use of co-researchers, please ensure financial statement/ breakdown is provided for that amount.*

**ii) If there is/are variation(s) between the proposed and actual expenditure, please provide the reasons:**

1. Vot 52000 was created for the insurance fee for the Toshiba Notebook bought under Vot 35000.

2. The two Mathematica licenses were budgeted under Vot 29000 but bought under Vot 35000.

The cost of the software is RM 8300.00.

**iii) If there is/are variation(s) between the proposed and actual expenditure, please indicate the corrective action you plan to take:**

The actual expenditure is kept less than the total proposed expenditure

## I. KEY PERFORMANCE INDICATORS ACHIEVEMENT

### i) Scientific Knowledge Creation

No. of Publications (with impact factors)
*(**Please list down all related publications since project started and attach the hardcopy)*

| No | Type of Publication | No. of publication/s | Title | Impact factor |
|---|---|---|---|---|
| 1 | **Journal**<br><br>i) International<br>*[High impact or cited in Science Citation index (SCI) or Current Contents (CC)]* | 1. J. Math. Phys<br><br>2. Int. J. Mod. Phys A | 1. Monopole-antimonopole and their Vortex Rings<br>2. Dyons of One Half Monopole Charge | 1. 1.192<br>2. 1.472 |
| | ii) Local | 3. Jurnal Fizik Malaysia | 3. A-M-A Monopole Configurations | 3. Nil |
| 2 | **Papers (seminar/ conference/ workshop/Inaugural Lecture/Keynote)**<br><br>i) International | Four | 1. A-M-A Monopole Configurations<br>2. Half-Integer Topological Charge Multimonopole<br>3. A Note on the Julia-Zee Dyon Solution<br>4. Dyons of One Half Monopole Charge | |
| | ii) Local | Two | 5. A Numerical Investigation of the MAP Solution of the SU(2) YMH Theory<br>6. Some Comments on the Monopole | |

| | | | | |
|---|---|---|---|---|
| 3 | **Chapter in Scientific books/ Monographs**<br><br>i) International | NIL | | |
| | ii) Local | | | |
| 4 | **Electronic Journal (Peer reviewed/ with impact factors)**<br><br>i) International | NIL | | |
| | ii) Local | | | |

**ii) Technology Creation**
*(Relevant documents to be attached)*

| No | Technology creation (NOT APPLICABLE) | |
|---|---|---|
| *1* | Major scientific discoveries & new inventions | i)<br>ii)<br>iii) |
| *2* | No. of patents filed | i)<br>ii)<br>iii) |
| *3* | No. of patents attained | i)<br>ii)<br>iii) |
| *4* | No. of technology platforms created and transferred | i)<br>ii)<br>iii) |
| *5* | No. of technology platforms acquired & applied | i)<br>ii)<br>iii) |

**J. ORGANISATIONAL OUTCOMES OF THE PROJECT** ((Please describe as specifically as possible the organizational benefits arising from the project and provide an assessment of their significance)

**i. Contribution of the project to expertise development**

**a. How did the project contribute to expertise?**

- [x] PhD degrees — How many: 1
- [ ] MSc degrees — How many: ______
- [ ] Research staff with new specialty — How many: ______
- [ ] Other, please specify: ______

**b. How significant is this expertise?**

- [x] One of the key areas of priority for Malaysia
- [ ] An important area, but not a priority one

## G) Human Capacity Building

**i) No. of researchers in the team (since project started)**
***(Professor/Assoc. Prof./Lecturers (Dr.)/PhD student/Masters student/Research officer/ Research Assistant)***

| Age and gender / Research member | No. of Researchers | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | <20 | | 21-30 | | 31-40 | | 41-50 | | 51-60 | | 61-70 | | 71-80 | | 81-90 | |
| | Male | Female | Male | Female | Male | Female | Male | Female | Male | Female | Male | Female | Male | Female | Male | Female |
| *1) Professor* | | | | | | | | | | | | | | | | |
| *2) Assoc. Professor* | | | | | | | | | | 1 | | | | | | |
| *3) Lecturer (Dr.)* | | | | | | | | | | | | | | | | |
| *4) PhD students* | | | 1 | | | | | | | | | | | | | |
| *5) Masters student* | | | | | | | | | | | | | | | | |
| *6) Research officer* | | | | | | | | | | | | | | | | |
| 6a) No. of PhD level = ........ | | | | | | | | | | | | | | | | |
| 6b) No. of Master's level = ........one | | | 1 | | | | | | | | | | | | | |
| 6c) No. of Undergraduate level = ....... | | | | | | | | | | | | | | | | |
| 6d) Others (please state[if any]) = ......... | | | | | | | | | | | | | | | | |
| *7) Research assistant* | | | | | | | | | | | | | | | | |
| *7a) No. of Undergraduate level =.........* | | | | | | | | | | | | | | | | |
| *7b) No. of diploma level = .............* | | | | | | | | | | | | | | | | |
| 7c) Others (please state[if any]) = ......... | | | | | | | | | | | | | | | | |

**ii) Disciplines of researchers within team :**

1) HIGH ENERGY THEORETICAL PHYSICS

2)

3)

**ii. Economic contribution of the project? (only fill if relevant) – NOT APPLICABLE**

**a. How has the economic contribution of the project materialized?**

- [ ] Cost savings
- [ ] Time savings
- [ ] Other, please specify: ______________________________

**b. How important is this economic contribution?**

- [ ] High economic contribution Value: RM ____________
- [ ] Medium economic contribution Value: RM ____________
- [ ] Low economic contribution Value: RM ____________

**c. When has this economic contribution materialized?**

- [ ] Already materialized
- [ ] Within months of project completion
- [ ] Within three years of project completion
- [ ] Expected in three years or more
- [ ] Unknown

**iii. Infrastructural contribution of the project – NOT APPLICABLE**

**a. What infrastructural contribution has the project had?**

- [ ] New equipment Value: RM ____________
- [ ] New/improved facility Investment: RM ____________
- [ ] New information networks
- [ ] Other, please specify: ______________________________

**b. How significant is this infrastructural contribution for the organization?**

- [ ] Not significant/does not leverage other projects
- [ ] Moderately significant
- [ ] Very significant/significantly leverages other projects

| | |
|---|---|
| iv. | **Contribution of the project to the organization's reputation** |
| a. | **How has the project contributed to increasing the reputation of the organization** |

- [x] Recognition as a Center of Excellence
- [ ] National award
- [ ] International award
- [ ] Demand for advisory services
- [ ] Invitations to give speeches on conferences
- [ ] Visits from other organizations
- [ ] Other, please specify:

| | |
|---|---|
| b. | **How important is the project's contribution to the organization's reputation?** |

- [ ] Not significant
- [ ] Moderately significant
- [x] Very significant

## K. NATIONAL IMPACTS OF THE PROJECT

| | |
|---|---|
| i. | **Contribution of the project to organizational linkages** |
| a.. | **Which kinds of linkages did the project create?** |

- [ ] Domestic industry linkages
- [ ] International industry linkages
- [x] Linkages with domestic research institutions, universities
- [x] Linkages with international research institutions, universities

| | |
|---|---|
| b. | **What is the nature of the linkages?** |

- [ ] Staff exchanges
- [ ] Inter-organizational project team
- [ ] Research contract with a commercial client
- [x] Informal consultation
- [ ] Other, please specify: ____________________

| | |
|---|---|
| **ii.** | **Social-economic contribution of the project** |
| **a.** | **Who are the direct customer/beneficiaries of the project output?** |

| Customers/beneficiaries: | Number: |
|---|---|
| Undergraduate Physics students | 200 |

**b.** **How has/will the socio-economic contribution of the project materialized?**

- [ ] Improvements in health
- [ ] Improvements in safety
- [ ] Improvements in the environment
- [ ] Improvements in energy consumption/supply
- [ ] Improvements in international relations
- [x] Other, please specify: Intellectual knowledge

**iii.** **How important is this socio-economic contribution?**

- [x] High social contribution
- [ ] Medium social contribution
- [ ] Low social contribution

**iv.** **When has/will this social contribution materialized?**

- [ ] Already materialized
- [x] Within three years of project completion
- [ ] Expected in three years or more
- [ ] Unknown

Signature: [signature]

Date: July 10, 2007

Official Stamp: PROF. MADYA ROSY TEH CHOOI GIM
Timbalan Dekan
(Akademik dan Pembangunan Pelajar)
Pusat Pengajian Sains Fizik
Universiti Sains Malaysia
11800 Pulau Pinang

Endorsed by:
Research Management Centre
Official Stamp:

H(i) 1. & H(ii) 1.



# A-M-A monopole configurations

Rosy Teh and Khai-Ming Wong
*School of Physics, Universiti Sains Malaysia, 11800 USM Penang, Malaysia*

(Received 25 July 2005)

We present exact axially symmetrical static monopole configurations of the SU(2) Yang-Mills-Higgs theory. These configurations has a antimonopole-monopole-antimonopole (A-M-A) chain, when the parameter, $m = 1,2,3,\ldots$. Vortex rings coexist with the A-M-A configurations when $m \geq 2$. The monopole in the A-M-A chain is the Wu-Yang type monopole and it possesses a broken Dirac-like string potential. This monopole which arises from a singularity in the Higgs field possesses structure whereas the two antimonopoles which arises from point zeros of the Higgs field are structureless.

## I. INTRODUCTION

The SU(2) Yang-Mills-Higgs (YMH) field theory, with the Higgs field in the adjoint representation possesses magnetic monopole, multimonopole, antimonopoles, and vortex rings solutions [1-5]. The 't Hooft-Polyakov monopole solution is numerically, spherically symmetrical [1]. Exact monopole and multimonopoles solutions exist in the Bogomol'nyi-Prasad-Sommerfield (BPS) limit [2] whilst outside the BPS limit, when the Higgs field potential is nonvanishing only numerical solutions are known. Numerical BPS axially symmetric vortex rings solutions have also been reported [3].

We have reported on a different type of BPS static monopole-antimonopole solution [4]. In this paper we would like to point out the Wu-Yang type monopole in the A-M-A configurations of Ref. [5] is a composite monopole that posseses structure. In this theory, the SU(2) YMH Lagrangian in 3+1 dimensions is given by

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}D^\mu\Phi^a D_\mu\Phi^a - \frac{1}{4}\lambda\left(\Phi^a\Phi^a - \frac{\mu^2}{\lambda}\right)^2, \tag{1}$$

$$D_\mu\Phi^a = \partial_\mu\Phi^a + g\varepsilon^{abc}A^b_\mu\Phi^c, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc}A^b_\mu A^c_\nu, \tag{2}$$

where $\mu$ is the Higgs field mass, $\lambda$ is the strength of the Higgs potential and $\mu/\sqrt{\lambda}$ is the vacuum expectation value of the Higgs field. Since the gauge field coupling constant g, can be scaled away, we set it to one without any loss of generality. The metric used is $g_{\mu\nu} = (-+++)$. The SU(2) internal group indices *a, b, c* run from 1 to 3 and the spatial indices are $\mu, \nu, a = 0,1,2,$ and 3 in Minkowski space. The equations of motion that follow from the Lagrangian are

$$D^\mu F^a_{\mu\nu} = \partial^\mu F^a_{\mu\nu} + \varepsilon^{abc}A^{b\mu}F^c_{\mu\nu} = \varepsilon^{abc}\Phi^b D_\nu\Phi^c, \quad D^\mu D_\mu\Phi^a = -\lambda\Phi^a\left(\Phi^b\Phi^b - \frac{\mu^2}{\lambda}\right). \tag{3}$$

The Abelian electromagnetic field tensor as proposed by 't Hooft [1] is

$$F_{\mu\nu} = \hat{\Phi}^a F^a_{\mu\nu} - \varepsilon^{abc}\hat{\Phi}^a D_\mu\hat{\Phi}^b D_\nu\hat{\Phi}^c = \partial_\mu A_\nu - \partial_\nu A_\mu - \varepsilon^{abc}\hat{\Phi}^a\partial_\mu\hat{\Phi}^b\partial_\nu\hat{\Phi}^c, \tag{4}$$

where $A_\mu = \hat{\Phi}^a A^a_\mu$, $\hat{\Phi}^a = \Phi^a/|\Phi|$, $|\Phi| = \sqrt{\Phi^a\Phi^a}$. Hence the 't Hooft electric field is $E_i = F_{0i}$, and the 't Hooft magnetic field is $B_i = -\frac{1}{2}\varepsilon_{ijk}F_{jk}$, where the indices, $i,j,k = 1,2,3$. The topological magnetic current, which is also the topological current density of the system is $k_\mu = \frac{1}{8\pi}\varepsilon_{\mu\nu\rho\sigma}\varepsilon_{abc}\partial^\nu\hat{\Phi}^a\partial^\rho\hat{\Phi}^b\partial^\sigma\hat{\Phi}^c$. Therefore the corresponding conserved topological magnetic charge is

$$M = \int d^3x\, k_0 = \frac{1}{8\pi}\oint d^2\sigma_i(\varepsilon_{ijk}\varepsilon^{abc}\hat{\Phi}^a\partial_j\hat{\Phi}^b\partial_k\hat{\Phi}^c) = \frac{1}{4\pi}\oint d^2\sigma_i\, B_i. \tag{5}$$

## II. THE ANSATZ AND ITS FORMULATION

The static gauge fields and Higgs field are given respectively by [5]

$$A_\mu^a = \frac{1}{r}\psi(r)(\hat{\theta}^a\hat{\phi}_\mu - \hat{\phi}^a\hat{\theta}_\mu) + \frac{1}{r}R(\theta)(\hat{\phi}^a\hat{r}_\mu - \hat{r}^a\hat{\phi}_\mu),$$
$$\Phi^a = \frac{1}{r}\psi(r)\,\hat{r}^a + \frac{1}{r}R(\theta)\,\hat{\theta}^a. \quad (6)$$

The spherical coordinate orthonormal unit vectors are defined by

$$\hat{r}^a = \sin\theta\cos\phi\,\delta_1^a + \sin\theta\sin\phi\,\delta_2^a + \cos\theta\,\delta_3^a,$$
$$\hat{\theta}^a = \cos\theta\cos\phi\,\delta_1^a + \cos\theta\sin\phi\,\delta_2^a - \sin\theta\,\delta_3^a,$$
$$\hat{\phi}^a = -\sin\theta\,\delta_1^a + \cos\phi\,\delta_2^a,$$

where
$r = \sqrt{x^i x_i}$, $\theta = \cos^{-1}(x_3/r)$, and $\phi = \tan^{-1}(x_2/x_1)$.
Ansatz (6) satisfies the Coulomb gauge, $\partial^i A_i^a = A_0^a = 0$. Upon substituting ansatz (6) into the Bogomol'nyi equations, $B_i^a \pm D_i\Phi^a = 0$, with the positive sign, the resulting equations are just two first order differential equations,

$$r\frac{\partial\psi}{\partial r} + \psi - \psi^2 = -m(m+1),$$
$$\frac{\partial R}{\partial\theta} + R\cot\theta - R^2 = m(m+1). \quad (7)$$

The solutions for $\psi$ and $R$ are then given respectively by

$$\psi = \frac{(m+1) - m(br)^{2m+1}}{1+(br)^{2m+1}},$$
$$R = (m+1)\left\{\cot\theta - \frac{(P_{m+1}(\cos\theta) + aQ_{m+1}(\cos\theta))}{(P_m(\cos\theta) + aQ_m\cos\theta))}\csc\theta\right\} \quad (8)$$

where $P_m$ and $Q_m$ are the Legendre polynomials of the first and second kind of degree $m$, respectively, and $m = 0,1,2,3,\ldots$. For solutions regular along the $z$-axis, we required, $R(0) = R(\pi) = 0$, and the integration constant $a$ = 0. The integration constant $b$ determines the position of the zeros of the Higgs field along the $z$-axis and without any loss in generality, we set $b$ = 1. The boundary conditions for $\psi(r)$ are $\psi(0) = m+1$ and $\psi(\infty) = -m$.

From the ansatz (6), $A_\mu = \hat{\Phi}^a A_\mu^a = 0$, hence the 't Hooft electric field is zero and the 't Hooft magnetic field is independent of the gauge fields $A_\mu^a$. To calculate for the 't Hooft magnetic field $B_i$, we rewrite the Higgs field from the spherical to the Cartesian coordinate system, [5]

$$\Phi^a = \Phi_1\hat{r}^a + \Phi_2\hat{\theta}^a + \Phi_3\hat{\phi}^a$$
$$= \tilde{\Phi}_1\delta^{a1} + \tilde{\Phi}_2\delta^{a2} + \tilde{\Phi}_3\delta^{a3}; \quad (9)$$

where

$$\tilde{\Phi}_1 = \sin\theta\cos\phi\,\Phi_1 + \cos\theta\cos\phi\,\Phi_2 - \sin\phi\,\Phi_3$$
$$= |\Phi|\cos\alpha\sin\beta,$$
$$\tilde{\Phi}_2 = \sin\theta\sin\phi\,\Phi_1 + \cos\theta\sin\phi\,\Phi_2 + \cos\phi\,\Phi_3$$
$$= |\Phi|\cos\alpha\cos\beta,$$
$$\tilde{\Phi}_3 = \cos\theta\,\Phi_1 - \sin\theta\,\Phi_2 = |\Phi|\sin\alpha.$$

The Higgs unit vector can be simplified to

$$\hat{\Phi}^a = \cos\alpha\sin\beta\,\delta^{a1} + \cos\alpha\cos\beta\,\delta^{a2} + \sin\alpha\,\delta^{a3}, \quad (10)$$

and the 't Hooft magnetic field is reduced to

$$B_i = -\frac{1}{r^2\sin\theta}\left\{\frac{\partial\sin\alpha}{\partial\theta}\right\}\hat{r}_i + \frac{1}{r\sin\theta}\left\{\frac{\partial\sin\alpha}{\partial r}\right\}\hat{\theta}_i, \quad (11)$$

where $\sin\alpha = \dfrac{\psi\cos\theta - R\sin\theta}{\sqrt{\psi^2 + R^2}}$, $\beta = \dfrac{\pi}{2} - \phi$.

Since $\sin\alpha$ is a nonsingular function except at the points where the Higgs field, $r\Phi^a$ vanishes, the 't Hooft magnetic field is regular everywhere except at the locations of the A-M-A and the vortex rings.

We also note that, $B_i = \varepsilon_{ijk}\partial^j(\sin\alpha)\,\partial^k\beta = \varepsilon_{ijk}\partial^j(\sin\alpha\,\partial^k\beta)$, and a suitable Maxwell four-vector gauge potential for this magnetic field is $\mathcal{A}_0 = 0$, $\mathcal{A}_i = (\sin\alpha - 1)\partial_i\beta = -\dfrac{(\sin\alpha - 1)}{r\sin\theta}\hat{\phi}_i$. When $m$ = 0, the Maxwell gauge potential is just the usual Dirac string potential and it is singular along the negative $z$-axis.

## III. THE A-M-A CONFIGURATIONS

The magnetic charge enclosed by the sphere at infinity, Eq. (5), is calculated to be, $M_\infty = -1$, when $m = 1,2,3,\ldots$ and $M_\infty = 1$, when $m$ = 0. Hence the magnetic charge of the system is always negative one when $m > 0$. However when the radius of the enclosing sphere shrinks to zero at the origin, the magnetic charge becomes $M_0 = 1$, $m = 0,1,2,3,\ldots$. In fact, it is true that for positive nonzero integer $m$, when the radius of the enclosing sphere is $r < \sqrt[2m+1]{\frac{m+1}{m}}$, the topological magnetic charge is one, and when $r > \sqrt[2m+1]{\frac{m+1}{m}}$, the topological magnetic charge is negative one. Hence along the $z$-axis, there is a monopole located at $r = 0$ and two antimonopoles located at $z = \pm \sqrt[2m+1]{\frac{m+1}{m}}$ giving rise to the axially symmetric A-M-A monopole configurations. The two antimonopoles arise from the point zeros of the Higgs field and they are structureless whereas the monopole arise from the point singularity of the Higgs field at the origin. This monopole is the Wu-Yang type monopole and it possesses structure.

Our studies reveal that the monopole at the origin corresponds to a zero size composite monopole. This Wu-Yang type monopole is composed of a string of A and M "poles" arranged vertically along the $z$-axis. By induction we conclude that the number of "poles" in the composite monopole is given by $2m+1$. When $m$ is even, the "pole" in the center of the structure is a M monopole. When $m$ is odd, we have a MAM monopole in the center of the axis. The other "poles" in the composite monopoles both above and below the M monopole and MAM monopole appear in pairs of MA. These "poles" have magnetic charge of less than unity and are not real "poles" but just vortex points. Hence the Wu-Yang type monopole has a corresponding structure of ...MAMAMAMAM... when $m$ is even and a corresponding structure of ...MAMAMAMAMAM... when $m$ is odd with the total number of "poles" being $2m+1$.

This monopole posseses the usual Dirac string potential in the Abelian gauge when $m = 0$. However in the A-M-A configuration, when $m = 1,2,3,\ldots$, the Dirac string is broken into two parts, a finite piece, $-\sqrt[2m+1]{(m+1)/m} < z < 0$, along the negative $z$-axis and a semi-infinite piece, $z > \sqrt[2m+1]{(m+1)/m}$, along the positive $z$-axis.

Horizontally positioned vortex rings start to appear around the $z$-axis, when the parameter $m$ exceeds unity. The number of vortex rings in the solution is equal to $(m-1)$. These vortex rings arise from the ring zeros of the Higgs field when $m \geq 2$. The magnetic field lines from the Wu-Yang type monopole pass through these vortex rings before going to infinity.

The singularities in $R(\theta)$ when $P_m(\cos\theta) = 0$ give rise to plane singularities. The number of singular planes in the solution is equal to $m$, hence corresponding to the number of A "poles" in the composite monopole. In all these solutions, the 't Hooft magnetic foelds possess negative Dirac delta function singularity along these planes as the gauge potentials are discontinuous at these values of $\theta$. These A-M-A configurations have zero magnetic dipole moment as the number of poles in these solutions is odd. Also for each A-M-A configuration, there always exist the anti-version, the M-A-M configuration.

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H(1) 2.

# Half-integer topological charge multimonopole

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It has been shown that the SU(2) YMH theory possess axially symmetrical monopole of half integer topological charge. In this paper, we would like to present exact SU(2) YMH monopole solutions of half-integer topological charge. These solutions can be just an isolated half-monopole or a multimonopole with topological magnetic charge $\frac{1}{2}m$ where $m$ is a natural number. These static monopole solutions satisfy the first order Bogomo'nyi.

The SU(2) Yang-Mills-Higgs (YMH) field theory in 3+1 dimensions, with the Higgs field adjoint representation, possess both the monopole and multimonopole solutions [1-3]. Exact monopole and multimonopole solutions exist only in the Bogomol'nyi-Prasad-Sommerfield (BPS) limit with vanishing Higgs potential [2,3]. In general, monopoles with unit magnetic charge are spherically symmetric but multimonopole configurations with magnetic charge greater than unity possess at most axial symmetry and cannot possess spherical symmetry [4].

Half-monopole is the configuration with one half of the usual 't Hooft unit magnetic charge. It has been shown that such a configuration exists in the SU(2) Yang-Mills theory [5] but they possess a Dirac-like string singularity along the negative $z$-axis in both the gauge potential as well as the Abelian magnetic field. These half-monopoles are all axially symmetric.

Recently we have presented in Ref. [6], the axially symmetric half-monopole and the C series solutions with only mirror symmetry along the $x$-$z$ plane. In this paper we will discuss further on the properties of the C series solutions.

The SU(2) YMH Lagrangian with vanishing Higgs mass and self interaction is

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}D^\mu\Phi^a D_\mu\Phi^a. \tag{1}$$

The Lagrangian (1) is gauge invariant under the set of independent local SU(2) transformations at each space-time point. The covariant derivative of the Higgs field and the gauge field strength tensor are given respectively by $D_\mu\Phi^a = \partial_\mu\Phi^a + g\varepsilon^{abc}A^b_\mu\Phi^c$ and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc}A^b_\mu A^c_\nu$. The gauge field coupling constant $g$ is set to one without any loss of generality. The metric used is $g_{\mu\nu} = (-+++)$. The SU(2) internal group indices $a$, $b$, $c$ run from 1 to 3 and the spatial indices are $\mu, \nu, a = 0,1,2$, and 3 in Minkowski space.

The equations of motion that follow from the Lagrangian (1) are

$$D^\mu F^a_{\mu\nu} = \varepsilon^{abc}\Phi^b D_\nu\Phi^c, \quad D^\mu D_\mu\Phi^a = 0. \tag{2}$$

The Bogomol'nyi equation is $B^a_i \pm D_i\Phi^a = 0$. The tensor identified with the Abelian electromagnetic field, as proposed by 't Hooft [1] is

$$\begin{aligned} F_{\mu\nu} &= \hat{\Phi}^a F^a_{\mu\nu} - \varepsilon^{abc}\hat{\Phi}^a D_\mu\hat{\Phi}^b D_\nu\hat{\Phi}^c \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - \varepsilon^{abc}\hat{\Phi}^a\partial_\mu\hat{\Phi}^b\partial_\nu\hat{\Phi}^c, \end{aligned} \tag{3}$$

where $A_\mu = \hat{\Phi}^a A^a_\mu$, $\hat{\Phi}^a = \Phi^a/|\Phi|$, $|\Phi| = \sqrt{\Phi^a\Phi^a}$. Hence the 't Hooft electric field is $E_i = F_{0i}$, and the 't Hooft magnetic field is $B_i = -\frac{1}{2}\varepsilon_{ijk}F_{jk}$, where the indices, $i, j, k = 1,2,3$. The topological magnetic current, which is also the topological current density of the system is $k_\mu = \frac{1}{8\pi}\varepsilon_{\mu\nu\rho\sigma}\varepsilon_{abc}\partial^\nu\hat{\Phi}^a\partial^\rho\hat{\Phi}^b\partial^\sigma\hat{\Phi}^c$ [7]. Therefore the corresponding conserved topological magnetic charge is

$$\begin{aligned} M &= \int d^3x\, k_0 = \frac{1}{8\pi}\int \varepsilon_{ijk}\varepsilon^{abc}\partial_i(\hat{\Phi}^a\partial_j\hat{\Phi}^b\partial_k\hat{\Phi}^c)d^3x \\ &= \frac{1}{8\pi}\oint d^2\sigma_i(\varepsilon_{ijk}\varepsilon^{abc}\partial_i(\hat{\Phi}^a\partial_j\hat{\Phi}^b\partial_k\hat{\Phi}^c) \\ &= \frac{1}{4\pi}\oint d^2\sigma_i\, B_i. \end{aligned} \tag{4}$$

The magnetic ansatz is given respectively by

$$\begin{aligned} A^a_\mu &= \frac{1}{r}\psi(r)(\hat{\theta}^a\hat{\phi}_\mu - \hat{\phi}^a\hat{\theta}_\mu) + \frac{1}{r}R(\theta)(\hat{\phi}^a\hat{r}_\mu - \hat{r}^a\hat{\phi}_\mu) \\ &\quad + \frac{1}{r}G(\theta,\phi)(\hat{r}^a\hat{\theta}_\mu - \hat{\theta}^a\hat{r}_\mu), \\ \Phi^a &= \frac{1}{r}\psi(r)\hat{r}^a + \frac{1}{r}R(\theta)\hat{\theta}^a + \frac{1}{r}G(\theta,\phi)\hat{\phi}^a. \end{aligned} \tag{5}$$

The spherical coordinate orthonormal unit vectors are defined by

$$\hat{r}^a = \sin\theta\cos\phi\,\delta_1^a + \sin\theta\sin\phi\,\delta_2^a + \cos\theta\,\delta_3^a,$$
$$\hat{\theta}^a = \cos\theta\cos\phi\,\delta_1^a + \cos\theta\sin\phi\,\delta_2^a - \sin\theta\,\delta_3^a, \qquad (6)$$
$$\hat{\phi}^a = -\sin\theta\,\delta_1^a + \cos\phi\,\delta_2^a,$$

where $r = \sqrt{x^i x_i}$, $\theta = \cos^{-1}(x_3/r)$, and $\phi = \tan^{-1}(x_2/x_1)$. The gauge fixing condition that we used here is the Coulomb gauge, $\partial^i A_i^a = 0$, $A_0^a = 0$.

From the ansatz (5), $A_\mu = \hat{\Phi}^a A_\mu^a = 0$. Hence the 't Hooft electric field is zero and the 't Hooft magnetic field is independent of the gauge fields $A_\mu^a$. To calculate for the 't Hooft magnetic field $B_i$, we rewrite the Higgs field of Eq. (5) from the spherical to the Cartesian coordinate system, [6]

$$\Phi^a = \Phi_1\hat{r}^a + \Phi_2\hat{\theta}^a + \Phi_3\hat{\phi}^a = \widetilde{\Phi}_1\delta^{a1} + \widetilde{\Phi}_2\delta^{a2} + \widetilde{\Phi}_3\delta^{a3} \qquad (7)$$

where

$$\begin{aligned}\widetilde{\Phi}_1 &= \sin\theta\cos\phi\,\Phi_1 + \cos\theta\cos\phi\,\Phi_2 - \sin\phi\,\Phi_3 \\ &= |\Phi|\cos\alpha\sin\beta, \\ \widetilde{\Phi}_2 &= \sin\theta\sin\phi\,\Phi_1 + \cos\theta\sin\phi\,\Phi_2 + \cos\phi\,\Phi_3 \\ &= |\Phi|\cos\alpha\cos\beta, \\ \widetilde{\Phi}_3 &= \cos\theta\,\Phi_1 - \sin\theta\,\Phi_2 = |\Phi|\sin\alpha.\end{aligned} \qquad (8)$$

The Higgs unit vector is then

$$\Phi^a = \cos\alpha\sin\beta\,\delta^{a1} + \cos\alpha\cos\beta\,\delta^{a2} + \sin\alpha\,\delta^{a3}, \qquad (9)$$

where, $\sin\alpha = \dfrac{\psi\cos\theta - R\sin\theta}{\sqrt{\psi^2 + R^2 + G^2}}$, $\beta = \gamma - \phi$, $\gamma = \tan^{-1}\left(\dfrac{\psi\sin\theta + R\cos\theta}{G}\right)$ and the 't Hooft magnetic field is found to be

$$\begin{aligned}B_i = &\frac{\csc\theta}{r^2}\left\{\frac{\partial\sin\alpha}{\partial\theta}\frac{\partial\beta}{\partial\phi} - \frac{\partial\sin\alpha}{\partial\phi}\frac{\partial\beta}{\partial\theta}\right\}\hat{r}_i \\ &+ \frac{\csc\theta}{r^2}\left\{\frac{\partial\sin\alpha}{\partial\phi}\frac{\partial\beta}{\partial r} - \frac{\partial\sin\alpha}{\partial r}\frac{\partial\beta}{\partial\phi}\right\}\hat{\theta}_i \\ &+ \frac{1}{r}\left\{\frac{\partial\sin\alpha}{\partial r}\frac{\partial\beta}{\partial\theta} - \frac{\partial\sin\alpha}{\partial\theta}\frac{\partial\beta}{\partial r}\right\}\hat{\phi}_i.\end{aligned} \qquad (10)$$

The ansatz (5) is solved with the positive sign Bogomol'nyi equations and the resulting equations of motion are just four coupled first order partial differential equations,

$$r\frac{\partial\psi}{\partial r} + \psi - \psi^2 = -p,\quad \frac{\partial R}{\partial\theta} + R\cot\theta - R^2 = p - m^2\csc^2\theta,$$
$$\frac{\partial G}{\partial\theta} + G\cot\theta = 0,\quad \frac{\partial G}{\partial\phi}\csc\theta - G^2 = m^2\csc^2\theta, \qquad (11)$$

where $p$ and $m$ are arbitary constants. The C series of solutions [6] is solved by setting $p = 0$ in Eqs. (11). The solutions obtained are

$$\psi(r) = \frac{1}{1+r},\quad R(\theta) = m\csc\theta,$$
$$G(\theta,\phi) = m\csc\theta\tan(m\phi), \qquad (12)$$

where $m$ is restricted to take half integer values for $G$ to be a single value function. The boundary conditions are then $\psi(r)\big|_{r\to 0} = 1$, $\psi(r)\big|_{r\to\infty} = 0$ and $G(\theta,0) = G(\theta,2\pi) = 0$. The C solution is a series of multimonopole solutions with half-integer topological magnetic charge. The multimonopole is located at the origin $r = 0$, and has positive topological charge $M = |m| \in \left\{\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2, \pm\frac{5}{2}, \ldots\right\}$. The sign of the parameter $m$ does not determine the sign of the magnetic charge $M$. It only inverts the orientation of the multimonopole in ordinary 3D space, vertically.

These multimonopoles are composed of half-monopoles arranged symmetrically about the *x-z* plane. The orientation of these half-monopoles in the C multimonopole can be pictured in Fig. 1. This figure shows the 3D surface plot of the Abelian magnetic energy density, $B_iB_i$, at small $r$ = 0.001 for the configurations when $m = \frac{1}{2}, 1, \frac{3}{2}$, and 2. The plane singularity in the solution $G$ acts as a separator between the half-monopoles. The magnetic field, $B_i = B_r\hat{r}_i + B_\theta\hat{\theta}_i + B_\phi\hat{\phi}_i$, is absolutely zero along these plane singularities, hence "cutting" the C multimonopole into half-monopoles. This is shown in Fig. 2 which is the 2D vector field plot along the *x-y* plane for the configurations when $m = \frac{1}{2}, 1, \frac{3}{2}$, and 2.

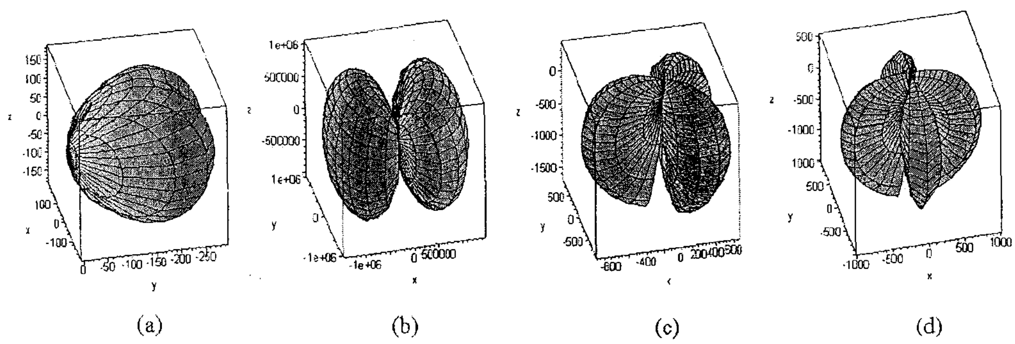


FIG. 1. (a) $m = \frac{1}{2}$, (b) $m = 1$, (c) $m = \frac{3}{2}$, and (d) $m = 2$.

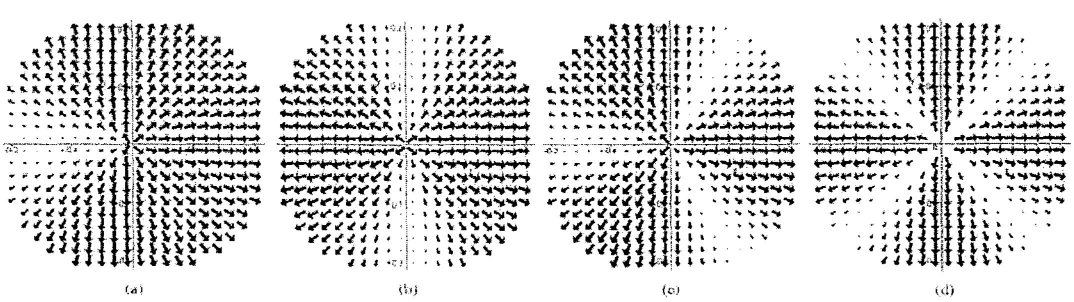


FIG. 2. (a) $m = \frac{1}{2}$, (b) $m = 1$, (c) $m = \frac{3}{2}$, and (d) $m = 2$.

From the C solution, we conclude that the most elementary unit of a monopole is one half of the charge of a 't Hooft-Polyakov monopole. This multimonopole which arises from a point singularity of the Higgs field at the origin is similar to the A-M-A configurations monopole [8] in that they posses structure and are composite monopoles. The C multimonopole consists of $2|m|$ half-monopoles.

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# A Note on the Julia-Zee Dyon Solution

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**Abstract**

We observed that the Julia-Zee dyon solution can be presented in similar exact form when the $\phi$-winding number of the internal space is $n$. However the closed form $n$-monopole version of the Julia-Zee dyon solution exist in the present of $(n-1)$ string antimonopoles. Hence the net monopole charge of the system at large distances is still unity. Using the ansatz of this solution, we also present the antimonopole version of the Julia-Zee dyon in closed form. We would also like to note that for a given monopole charge in this dyon solution, the net electric charge of the system can be both positive and negative.

## 1 Introduction

The SU(2) Yang-Mills-Higgs (YMH) field theory in $3+1$ dimensions, with the Higgs field in the adjoint representation possess a large varieties of magnetic monopole configurations and hence dyon solutions. The 't Hooft-Polyakov monopole solution [1] of this theory is invariant under a U(1) subgroup of the local SU(2) gauge group. In 1975, Julia and Zee [2] came up with a more general ansatz to construct the dyon solution from the given 't Hooft-Polyakov monopole solution. Both these solutions are numerical solutions. Later, Prasad and Sommerfield [3] gave the closed form version of both these solutions. After Prasad and Sommerfield's work, Protogenov [4] presented the general form of the exact solution with two extra parameters. However this general solution was presented for the pure Yang-Mill sourceless equation.

There were various works that had attempted to generalize the Julia-Zee dyon solution, one of which was the paper by I. Ju [5]. However this paper had several flaws. For example, the solution Eq.(15) given in their paper possessed a parameter $\delta$ which was not specified. Their solution Eq.(35c) is not a solution of the YMH equations.

Recently, we have also shown that the ansatz of Ref.[6] possesses more exact multimonopole-antimonopole configurations in the BPS limit. We have also constructed the anti-configurations of all these multimonopole-antimonopole solutions [7] with all the magnetic monopole charges reversing their sign. Hence monopoles becomes antimonopoles and vice versa.

Although the mathematics involved in the calculation of the dyon solution from a given monopole solution is standard in the SU(2) YMH field theory, the introduction of electric charge into the monopole systems is interesting and worth studying as it leads to more new physics. Also an exact BPS solution will become non-BPS with the introduction of an electric field in the monopole system.

In this paper we would like to give some remarks on the exact Julia-Zee dyon solution [2]. We presented a $n$-monopole charge Julia-Zee dyon solution. However this closed form $n$-monopole version of the Julia-Zee dyon solution exist only in the present of $(n-1)$ Dirac string antimonopoles which enter into the system in the form of a pure gauge potential linearly superposed with the non-Abelian gauge potential. Hence the net monopole charge of the system is still unity at infinity. The relationship between the non-Abelian monopoles or (zeros of the Higgs field) and the Dirac string monopoles had been beautifully described by the work of Arafune et al. [8]. When $n=1$, the solution is just the Julia-Zee dyon solution.

We also present the antimonopole version of the Julia-Zee dyon in closed form. The procedure involved has been the same as those given in Ref.[7] for converting the multimonopole-antimonopoles configurations into the multi-antimonopole-monopoles configurations. Our study shows that with a given positive or negative monopole charge, the corresponding Julia-Zee dyon solutions can possess both positive or negative net electric charge, as to every given gauge potentials, the Higgs field can take on both $\pm$ signs. We would also like to comment on the general statement made by Ref. [9] that the $\pm$ sign of the Bogomol'nyi equations, $(B_i^a \pm D_i\Phi^a) = 0$, corresponds to monopoles and antimonopoles respectively is not true. It is the $\phi$-winding number of the internal space that will have to be a positive or negative one in order to obtain a monopole or an antimonopole solution respectively.

## 2 The SU(2) YMH Theory

The SU(2) YMH Lagrangian in 3+1 dimensions with non vanishing Higgs potential is given by

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}D^\mu\Phi^a D_\mu\Phi^a - \frac{1}{4}\lambda(\Phi^a\Phi^a - \frac{\mu^2}{\lambda})^2. \quad (1)$$

Here the Higgs field mass is $\mu$ and the strength of the Higgs potential is $\lambda$ which are constants. The vacuum expectation value of the Higgs field is $\mu/\sqrt{\lambda}$. The Lagrangian (1) is gauge invariant under the set of independent local SU(2) transformations at each space-time point. The covariant derivative of the Higgs field and the gauge field strength tensor are given respectively by

$$D_\mu\Phi^a = \partial_\mu\Phi^a + g\epsilon^{abc}A^b_\mu\Phi^c, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu. \quad (2)$$

Since the gauge field coupling constant $g$ can be scaled away, we can set $g$ to one without any loss of generality. The metric used is $g_{\mu\nu} = (- + ++)$. The SU(2) internal group indices $a, b, c$ run from 1 to 3 and the spatial indices are $\mu, \nu, \alpha = 0, 1, 2,$ and 3 in Minkowski space.

The equations of motion that follow from the Lagrangian (1) are

$$D^\mu F^a_{\mu\nu} = \epsilon^{abc}\Phi^b D_\nu \Phi^c, \quad D^\mu D_\mu \Phi^a = -\lambda\Phi^a(\Phi^b\Phi^b - \frac{\mu^2}{\lambda}). \tag{3}$$

The Bogomol'nyi equations which holds in the limit of vanishing $\mu$ and $\lambda$ is

$$B^a_i \pm D_i\Phi^a = 0. \tag{4}$$

In the case of the exact BPS 't Hooft-Polyakov monopole solution, the $\pm$ sign does not correspond to monopoles and antimonopoles respectively as was stated in Ref.[9]. Instead the $\pm$ sign corresponds to a change in sign of the Higgs field. When the electric field is switched on, the changed in sign of the Higgs field will correspond to a change in sign of the electric field and hence the electric charge of the dyon solutions. In the case of Ref.[6], the multimonopole-antimonopoles are solved with the $+$ sign, whereas the corresponding anti-multimonopole-monopole solutions [7] are solvable with the $-$ sign of the Bogomol'nyi equations together with a negative one $\phi$-winding number of the internal space.

The Abelian electromagnetic field proposed by 't Hooft [1] is

$$F_{\mu\nu} = \hat{\Phi}^a F^a_{\mu\nu} - \epsilon^{abc}\hat{\Phi}^a D_\mu\hat{\Phi}^b D_\nu\hat{\Phi}^c = \partial_\mu A_\nu - \partial_\nu A_\mu - \epsilon^{abc}\hat{\Phi}^a\partial_\mu\hat{\Phi}^b\partial_\nu\hat{\Phi}^c, \tag{5}$$

where $A_\mu = \hat{\Phi}^a A^a_\mu$, the Higgs unit vector, $\hat{\Phi}^a = \Phi^a/|\Phi|$, and the Higgs field magnitude $|\Phi| = \sqrt{\Phi^a\Phi^a}$. The Abelian electric field is $E_i = F_{0i}$, and the Abelian magnetic field is $B_i = -\frac{1}{2}\epsilon_{ijk}F_{jk}$. We would also like to write the Abelian 't Hooft electromagnetic field as

$$F_{\mu\nu} = M_{\mu\nu} + H_{\mu\nu}, \tag{6}$$

$$\text{where} \quad M_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad H_{\mu\nu} = -\epsilon^{abc}\hat{\Phi}^a\partial_\mu\hat{\Phi}^b\partial_\nu\hat{\Phi}^c, \tag{7}$$

which we refer to as the gauge part and the Higgs part of the 't Hooft electomagnetic field respectively.

The topological magnetic current [10] which is also the topological current density of the system and the corresponding conserved topological magnetic charge are respectively

$$k_\mu = \frac{1}{8\pi}\,\epsilon_{\mu\nu\rho\sigma}\,\epsilon_{abc}\,\partial^\nu\hat{\Phi}^a\,\partial^\rho\hat{\Phi}^b\,\partial^\sigma\hat{\Phi}^c, \tag{8}$$

$$M = \int d^3x\, k_0 = \frac{1}{8\pi}\oint d^2\sigma_i\left(\epsilon_{ijk}\epsilon^{abc}\hat{\Phi}^a\partial_j\hat{\Phi}^b\partial_k\hat{\Phi}^c\right). \tag{9}$$

As mentioned by Arafune et al. [8], the magnetic charge $M$ is the total magnetic charge of the system if and only if the gauge field is non singular. If the gauge field is singular and carries Dirac string monopoles, then the total magnetic charge of the system is the total sum of the Dirac string monopoles and the monopoles carry by the Higgs field which is $M$.

## 3 Solution with Positive $\phi$-Winding Number

The magnetic ansatz of Ref.[6] can be further generalized in the standard way to accommodate for the presence of an electric field by writing

$$
\begin{aligned}
A_0^a &= \sinh\gamma\left(\frac{1}{r}\chi_1(r)\hat{u}_r^a + \frac{1}{r}\chi_2(\theta)\hat{u}_\theta^a\right)\\
A_i^a &= -\frac{1}{r}\psi_1(r)\hat{u}_\phi^a\hat{\theta}_i + \frac{n}{r}\psi_2(r)\hat{u}_\theta^a\hat{\phi}_i + \frac{1}{r}R_1(\theta)\hat{u}_\phi^a\hat{r}_i - \frac{n}{r}R_2(\theta)\hat{u}_r^a\hat{\phi}_i\\
\Phi^a &= \cosh\gamma\left(\frac{1}{r}\chi_1(r)\;\hat{u}_r^a + \frac{1}{r}\chi_2(\theta)\hat{u}_\theta^a\right),
\end{aligned}
\tag{10}
$$

where $\gamma$ is an arbitrary constant. The spherical coordinate orthonormal unit vectors, $\hat{r}_i$, $\hat{\theta}_i$, and $\hat{\phi}_i$ are defined by

$$
\begin{aligned}
\hat{r}_i &= \sin\theta\;\cos\phi\;\delta_{i1} + \sin\theta\;\sin\phi\;\delta_{i2} + \cos\theta\;\delta_{i3},\\
\hat{\theta}_i &= \cos\theta\;\cos\phi\;\delta_{i1} + \cos\theta\;\sin\phi\;\delta_{i2} - \sin\theta\;\delta_{i3},\\
\hat{\phi}_i &= -\sin\phi\;\delta_{i1} + \cos\phi\;\delta_{i2}.
\end{aligned}
\tag{11}
$$

and the isospin coordinate orthonormal unit vectors, $\hat{u}_r^a$, $\hat{u}_\theta^a$, and $\hat{u}_\phi^a$ are defined by

$$
\begin{aligned}
\hat{u}_r^a &= \sin\theta\;\cos n\phi\;\delta_1^a + \sin\theta\;\sin n\phi\;\delta_2^a + \cos\theta\;\delta_3^a,\\
\hat{u}_\theta^a &= \cos\theta\;\cos n\phi\;\delta_1^a + \cos\theta\;\sin n\phi\;\delta_2^a - \sin\theta\;\delta_3^a,\\
\hat{u}_\phi^a &= -\sin n\phi\;\delta_1^a + \cos n\phi\;\delta_2^a.
\end{aligned}
\tag{12}
$$

To solve for solutions, the ansatz (10) is substituted into the equations of motion (3) in the limit of vanishing $\lambda$, $\mu$ and $\mu^2/\lambda \to$ constant. We find that when we set

$$
\begin{aligned}
&\psi_1(r) = \psi(r),\quad \chi_1(r) = \chi(r),\quad R_1(\theta) = \chi_2(\theta) = 0,\\
&n\psi_2(r) = (n-1) + \psi(r),\quad nR_2(\theta) = (n-1)\cot\theta,
\end{aligned}
\tag{13}
$$

the equations of motion can be simplified to the two usual second order differential equations, where prime means $\frac{d}{dr}$,

$$
r^2\psi'' - (1-\psi)(2\psi - \psi^2 - \chi^2) = 0,\quad r^2\chi'' - 2\chi(1-\psi)^2 = 0. \tag{14}
$$

Upon integration, we get two sets of first order coupled non linear differential equations,

$$
r\psi' \pm \chi(1-\psi) = 0,\quad r\chi' - \chi \pm 2\psi \mp \psi^2 = 0, \tag{15}
$$

and the general solutions are [4]

$$
\begin{aligned}
\psi(r) = \psi_\pm &= 1 \pm \frac{\beta r}{(\sqrt{a^2+1}\sinh(\beta r) + a\cosh(\beta r))}\\
\chi(r) = \chi_\pm &= \pm\left(1 - \beta r\frac{(\sqrt{a^2+1}\cosh(\beta r) + a\sinh(\beta r))}{(\sqrt{a^2+1}\sinh(\beta r) + a\cosh(\beta r))}\right).
\end{aligned}
\tag{16}
$$

Here $a$ and $\beta$ are arbitrary constants. There are all together four different sets of solutions in Eq.(16). When $a$ is set to zero, solutions (16) become the exact Julia-Zee dyon solutions [3] and when $a = i$, they become the complex infinite energy dyon solutions discussed by Singleton [11]. The parameter $\beta$ determines the expectation value of the Higgs field as, $\Phi^a \rightarrow \mp(\beta\cosh\gamma)\hat{u}_r^a$, at $r$ infinity.

With Eq.(13), the gauge potentials (10) reduce to,

$$\begin{aligned} A_0^a &= \sinh\gamma\frac{1}{r}\chi(r)\,\hat{u}_r^a, \quad \Phi^a = \cosh\gamma\frac{1}{r}\chi(r)\,\hat{u}_r^a, \\ A_i^a &= \frac{1}{r}\psi(r)(\hat{u}_\theta^a\hat{\phi}_i - \hat{u}_\phi^a\hat{\theta}_i) + \frac{(n-1)}{r}\{\hat{u}_\theta^a - \cot\theta\,\hat{u}_r^a\}\hat{\phi}_i, \end{aligned} \quad (17)$$

which becomes the Julia-Zee dyon ansatz when $n = 1$. The $\phi$-winding number, $n$, in the gauge potentials (17) is restricted to only positive integer. Hence the solutions obtained from this ansatz possess positive magnetic charge. We also notice that the pure gauge potential term of $A_i^a$ in Eq.(17),

$$A_i^a(\text{pure}) = \frac{(n-1)}{r}\{\hat{u}_\theta^a - \cot\theta\,\hat{u}_r^a\}\hat{\phi}_i = -(n-1)\delta_3^a\partial_i\phi, \quad (18)$$

possesses a line singularity. As this singular gauge potential is a pure gauge, it does not appear in the non-Abelian magnetic and the electric field,

$$\begin{aligned} B_i^a &= \frac{1}{r^2}\{r\psi'(\hat{u}_\theta^a\hat{\theta}_i + \hat{u}_\phi^a\hat{\phi}_i) + \psi(2-\psi)\hat{u}_r^a\hat{r}_i\}, \\ E_i^a &= -\frac{1}{r^2}\sinh\gamma\{(r\chi' - \chi)\hat{u}_r^a\hat{r}_i + \chi(1-\psi)(\hat{u}_\theta^a\hat{\theta}_i + \hat{u}_\phi^a\hat{\phi}_i)\}. \end{aligned} \quad (19)$$

The energy or mass of the system,

$$\begin{aligned} \mathcal{E} &= \int dx^3(\frac{1}{2}B_i^aB_i^a + \frac{1}{2}E_i^aE_i^a + \frac{1}{2}D_i\Phi^aD_i\Phi^a) \\ &= \frac{4\pi}{r}\cosh^2\gamma\,\chi(r\chi' - \chi)\Big|_{r=0}^{r=\infty}. \end{aligned} \quad (20)$$

is finite, $\mathcal{E} = 4\pi\beta\cosh^2\gamma$, only when $a = 0$.

From the ansatz (10), the Abelian 't Hooft gauge potentials, magnetic and electric fields are respectively given by

$$A_0 = \sinh\gamma\frac{1}{r}\chi(r), \quad A_i = \hat{\Phi}^aA_i^a = -(n-1)\cos\theta\,\partial_i\phi, \quad (21)$$

$$B_i = M_i + H_i = -\frac{(n-1)}{r^2}\hat{r}_i + \frac{n}{r^2}\hat{r}_i, \quad E_i = -\frac{1}{r^2}\sinh\gamma(r\chi' - \chi)\hat{r}_i. \quad (22)$$

The 't Hooft magnetic field in Eq.(22) always corresponds to that of a unit monopole field irrespective of the values of $n$ and $a$ in the solutions. When $n \neq 1$, the configuration (17) can be viewed as that of a $n$-monopole in the Higgs field superposed with a $(n-1)$ string antimonopole in the pure gauge field [8]. Therefore the net magnetic charge of the system is one at large $r$.

Of course when $n = 1$, the string antimonopole disappear. When $a = 0$, the monopole in the Higgs field is the finite energy 't Hooft-Polyakov monopole. However when $a \neq 0$, the solution is singular at $r = 0$ and the monopole in the Higgs field becomes a Wu-Yang type monopole with infinite energy.

Similary, the 't Hooft electric field is non singular at the origin, $r = 0$, when $a = 0$. However when $a$ takes value other than zero, a point singularity exist at $r = 0$. This can be seen by calculating for the total electric charge,

$$Q = \sinh\gamma\{-4\pi(r\chi' - \chi)\delta(r) \pm 4\pi\psi(2-\psi)|_{r=0}^{r=\infty}\} = \pm 4\pi \sinh\gamma, \quad (23)$$

for all values of $a$. The net electric charge comes from the delta function point source when $a \neq 0$ and when $a = 0$, there is no delta function point source and the net charge is from the regular electric charge distribution, that is the second term of Eq.(23). We also notice that the net electric charge can be both positive or negative in a positive monopole charge field.

## 4 Solution with Negative $\phi$-Winding Number

Using the same procedure as in Ref.[7], we can construct the antimonopole version of the Julia-Zee dyon by setting

$$\begin{aligned} &\psi_1(r) = \psi(r), \quad \chi_1(r) = \chi(r), \quad R_1(\theta) = \chi_2(\theta) = 0, \\ &n\psi_2(r) = (n+1) - \psi(r), \quad nR_2(\theta) = (n+1)\cot\theta, \end{aligned} \quad (24)$$

in the ansatz (10). The gauge potentials of the anti-configuration are

$$\begin{aligned} A_0^a &= \sinh\gamma\frac{1}{r}\chi(r)\,\hat{u}_r^a, \quad \Phi^a = \cosh\gamma\frac{1}{r}\chi(r)\,\hat{u}_r^a, \\ A_i^a &= -\frac{1}{r}\psi(r)(\hat{u}_\theta^a\hat{\phi}_i + \hat{u}_\phi^a\hat{\theta}_i) + \frac{(n+1)}{r}\{\hat{u}_\theta^a - \cot\theta\,\hat{u}_r^a\}\hat{\phi}_i, \end{aligned} \quad (25)$$

$$\text{and } A_i^a(\text{pure}) = \frac{(n+1)}{r}\{\hat{u}_\theta^a - \cot\theta\hat{u}_r^a\}\hat{\phi}_i = -(n+1)\delta_3^a\partial_i\phi. \quad (26)$$

Substituting the solution Eq.(24) into the equations of motion (3) in the BPS limit will give the same two coupled second order differential equations for $\psi$ and $\chi$ as in Eq.(14). Hence the solutions (16) apply for the gauge potentials and scalar Higgs field of Eq.(25). In this case, the $\phi$-winding number, $n$, in solution (24) is restricted to only negative integers and the net monopole charge of the configuration is negative one irrespective of the value of $n$. When $n = -1$, the pure gauge term (26) disappears and the solution is the regular Julia-Zee dyon solution with a negative one 't Hooft monopole charge or in short the JZ anti-dyon solution.

Calculating for the non-Abelian magnetic and the electric field from the gauge potentials (25), we get

$$\begin{aligned} B_i^a &= \frac{1}{r^2}\{r\psi'(-\hat{u}_\theta^a\hat{\theta}_i + \hat{u}_\phi^a\hat{\phi}_i) - \psi(2-\psi)\hat{u}_r^a\hat{r}_i\} \\ E_i^a &= -\frac{1}{r^2}\sinh\gamma\{(r\chi' - \chi)\hat{u}_r^a\hat{r}_i + \chi(1-\psi)(\hat{u}_\theta^a\hat{\theta}_i - \hat{u}_\phi^a\hat{\phi}_i)\}, \end{aligned} \quad (27)$$

which is the exact JZ anti-dyon magnetic and electric field in the negative $n$ $\phi$-winding number internal space. The electromagnetic fields (27) are regular when $a = 0$. When $a > 0$, a point singularity exits at $r = 0$ in the solution and the 't Hooft-Polyakov antimonopole becomes the Wu-Yang type antimonopole. When $a < 0$, in addition to the point singularity at $r = 0$, a spherical shell singuarity is also present at $\tanh(\beta r) = -\frac{a}{\sqrt{a^2+1}}$. Similarly, the energy of the anti-dyon system is finite, $\mathcal{E} = 4\pi\beta\cosh^2\gamma$, when $a = 0$, for all negative values of $n$.

From the ansatz (10), the Abelian 't Hooft gauge potentials, magnetic and electric fields are respectively,

$$A_0 = \sinh\gamma\frac{1}{r}\chi(r), \quad A_i = \hat{\Phi}^a A_i^a = -(n+1)\cos\theta\,\partial_i\phi, \tag{28}$$

$$B_i = M_i + H_i = -\frac{(n+1)}{r^2}\hat{r}_i + \frac{n}{r^2}\hat{r}_i, \quad E_i = -\frac{1}{r^2}\sinh\gamma(r\chi' - \chi)\hat{r}_i. \tag{29}$$

The 't Hooft magnetic field in Eq.(29) always corresponds to that of a unit antimonopole field irrespective of the values of $n$ and $a$ in the solutions. When $n = -1$, the string monopole disappear and the solution is regular over all space. When $n \neq -1$, the configuration (25) can be viewed as that of a $|n|$-antimonopole in the Higgs field superposed with a $|n+1|$ string monopole in the pure gauge field [8]. Therefore the net magnetic charge of the system is negative one at large $r$. When $a = 0$, the monopole in the Higgs field is the finite energy 't Hooft-Polyakov antimonopole. However when $a \neq 0$, the pole in the Higgs field becomes a Wu-Yang type antimonopole with infinite energy.

Similary, the 't Hooft electric field is non singular at the origin, $r = 0$, when $a = 0$. However when $a$ takes value other than zero, a point singularity exists at $r = 0$. The net electric charge is given by,

$$Q = \sinh\gamma\{-4\pi(r\chi' - \chi)\delta(r) \pm 4\pi\psi(2-\psi)|_{r=0}^{r=\infty}\} = \pm 4\pi\sinh\gamma, \tag{30}$$

and it comes from the delta function point source when $a \neq 0$ and when $a = 0$, there is no delta function point source and the net charge is from the regular electric charge distribution, that is the second term of Eq.(30). We also notice that the net electric charge can be both positive or negative in the presence of a negative monopole charge field.

## 5 Comments

When solving the equations of motion (3), for solutions in the BPS limit, the $\pm$ sign of the Bogolmol'nyi equations (4) is not significant. The second order coupled differential equations will always reduced to two different sets of first order coupled differential equations upon integration corresponding to the two sets of equations, $B_i^a \pm D_i\Phi^a$. The $\pm$ sign will only give a change in the sign of the corresponding Higgs field. When the electric field is switched on to give a dyon solution and $\gamma \neq 0$, the change in sign of the Higgs field will give a change in sign of the electric field and hence the electric charge present in the fields. The

$\pm$ sign in the Bogomol'nyi equations will not change the sign of the monopole charge present in the fields. It is only the sign of the $\phi$-winding number, $n$, that will determines the presence of monopoles or antimonopoles.

Finally, we would like to conjecture that a monopole is always located at a point in the Higgs field where the Higgs unit vector $\hat{\Phi}^a$ becomes indeterminate. This point could be a point singularity in the Higgs field or a zero of the Higgs field. We also observed that a point singularity in the Higgs field corresponds to the presence of a Wu-Yang type monopole and a point zero of the Higgs field corresponds to the presence of a 't Hooft-Polyakov monopole of finite energy.

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H(i) 4, & H(ii) 2.







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**Abstract:** We would like to present some exact SU(2) Yang–Mills–Higgs dyon solutions of one-half monopole charge. These static dyon solutions satisfy the first order Bogomol'nyi equations and are characterized by a parameter, m. They are axially symmetric. The gauge potentials and the electromagnetic fields possess a string singularity along the negative z-axis and hence they possess infinite energy density along the line singularity. However the net electric charges of these dyons which varies with the parameter m are finite.

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# DYONS OF ONE-HALF MONOPOLE CHARGE*

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We would like to present some exact SU(2) Yang–Mills–Higgs dyon solutions of one-half monopole charge. These static dyon solutions satisfy the first order Bogomol'nyi equations and are characterized by a parameter, $m$. They are axially symmetric. The gauge potentials and the electromagnetic fields possess a string singularity along the negative $z$-axis and hence they possess infinite energy density along the line singularity. However the net electric charges of these dyons which varies with the parameter $m$ are finite.

*Keywords*: Please provide keywords.

## 1. Introduction

The SU(2) Yang–Mills–Higgs (YMH) field theory in 3+1 dimensions, with the Higgs field in the adjoint representation possess the magnetic monopoles solution.[1–13] The 't Hooft–Polyakov monopole solution with nonzero Higgs mass and self-interaction is the first monopole solution that possess finite energy. This numerical, spherically symmetric monopole solution of unit magnetic charge is invariant under a U(1) subgroup of the local SU(2) gauge group.[1–4]

Configurations of the YMH field theory with a unit magnetic charge are in general spherically symmetric[1–7] although exceptions do exist.[14] All other monopole configurations with magnetic charges greater than unity possess at most axial symmetry[8–13] and it has been shown that these solutions cannot possess spherical symmetry when the energy is finite.[15]

Until now, exact monopoles and multimonopoles solutions exist only in the Bogomol'nyi–Prasad–Sommerfield (BPS) limit.[6–13] Outside of this limit, where the Higgs field potential is nonvanishing only numerical solutions exist. Asymmetric multimonopole solutions with no rotational symmetry are all numerical solutions even in the BPS limit.[16,17]

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Recently, we have also shown that the extended ansatz of Refs. 18–20 possesses more exact multimonopole–antimonopole configurations. The anticonfigurations of all these multimonopole–antimonopole solutions with all the magnetic monopole charges reversing their sign were also constructed.[21] Hence monopoles becomes antimonopoles and vice versa. Solutions with vortex rings in the presence of an antimonopole–monopole–antimonopole (A-M-A) chain were also discussed.[22] We have also shown the existence of half-integer topological monopole charge solutions. This configuration possesses axial symmetry and a Dirac-like string singularity along the negative $z$-axis.[14] The existence of smooth Yang–Mills potentials which correspond to monopoles and vortices of one-half winding number has been demonstrated in Ref. 23 but no exact or numerical solutions of the YMH equations have been given.

In this paper, we would like to reexamine the axially symmetric one-half monopole solutions of the Ref. 14 once again in more detail by introducing electric charge to the system, hence creating what is called a dyon solution with a one-half monopole charge. The procedure of introducing an electric charge to a monopole solution is standard and was first shown by Julia and Zee in 1975.[24]

The one-half monopole that was presented in Ref. 14 is a particular case when the solution parameter $m = -1/2$. In this paper, we would like to show that the one-half monopole solution actually possesses a continuous parameter $m$, where $-1/2 \leq m < 0$. The Higgs field of this one-half monopole solution does not possess any point zeros and hence there are no other monopoles in the configuration except the one-half monopole which is located at $r = 0$, where the Higgs unit vector, $\hat{\Phi}^a$, becomes indeterminate and the Higgs field is singular.

By applying Gauss' law to the electric field of this dyon solution at large $r$, we find that the electric flux seems to come from a net electric charge of decreasing magnitude as $m$ approaches zero. When $m = 0$, the dyon solution of one-half monopole charge becomes the dyon of the Wu–Yang type monopole[18–20] with unit magnetic charge and zero net electric charge.

We briefly review the SU(2) YMH field theory in the next section and introduce the modified axially symmetric magnetic ansatz[18–20] in Sec. 3. We also discuss the dyon solutions of one-half monopole charge and their electromagnetic properties in Sec. 3. We end with some comments in Sec. 4.

## 2. The SU(2) Yang–Mills–Higgs Theory

The SU(2) YMH Lagrangian in $3+1$ dimensions is given by

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}D^\mu\Phi^a D_\mu\Phi^a - \frac{1}{4}\lambda(\Phi^a\Phi^a - \frac{\mu^2}{\lambda})^2\,, \tag{1}$$

where $\mu$ is the Higgs field mass, $\lambda$ is the strength of the Higgs potential and $\mu/\sqrt{\lambda}$ is the vacuum expectation value of the Higgs field. The Lagrangian (1) is gauge invariant under the set of independent local SU(2) transformations at each space–time point. The covariant derivative of the Higgs field and the gauge field strength tensor are given respectively by

$$D_\mu \Phi^a = \partial_\mu \Phi^a + \epsilon^{abc} A_\mu^b \Phi^c \,, \tag{2}$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc} A_\mu^b A_\nu^c \,. \tag{3}$$

Since the gauge field coupling constant, can be scaled away, we set it to one without any loss of generality. The metric used is $g_{\mu\nu} = (-+++)$. The SU(2) internal group indices $a$, $b$, $c$ run from 1 to 3 and the spatial indices are $\mu$, $\nu$, $\alpha = 0, 1, 2$, and 3 in Minkowski space.

The equations of motion that follow from the Lagrangian (1) are

$$\begin{aligned} D^\mu F_{\mu\nu}^a &= \partial^\mu F_{\mu\nu}^a + \epsilon^{abc} A^{b\mu} F_{\mu\nu}^c = \epsilon^{abc} \Phi^b D_\nu \Phi^c \,, \\ D^\mu D_\mu \Phi^a &= -\lambda \Phi^a \left( \Phi^b \Phi^b - \frac{\mu^2}{\lambda} \right) . \end{aligned} \tag{4}$$

Non-BPS solutions to the YMH theory are obtained by solving the second order differential equations of motion (4), whereas BPS solutions can be more easily obtained by solving the Bogomol'nyi equations,

$$B_i^a \pm D_i \Phi^a = 0 \,, \tag{5}$$

which is of first order.

The Abelian electromagnetic field tensor as proposed by 't Hooft[1] is

$$\begin{aligned} F_{\mu\nu} &= \hat{\Phi}^a F_{\mu\nu}^a - \epsilon^{abc} \hat{\Phi}^a D_\mu \hat{\Phi}^b D_\nu \hat{\Phi}^c \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - \epsilon^{abc} \hat{\Phi}^a \partial_\mu \hat{\Phi}^b \partial_\nu \hat{\Phi}^c \,, \end{aligned} \tag{6}$$

where $A_\mu = \hat{\Phi}^a A_\mu^a$, $\hat{\Phi}^a = \Phi^a / |\Phi|$, $|\Phi| = \sqrt{\Phi^a \Phi^a}$. Hence the 't Hooft electric field is $E_i = F_{0i}$, and the 't Hooft magnetic field is $B_i = -\frac{1}{2} \epsilon_{ijk} F_{jk}$, where the indices, $i$, $j$, $k = 1, 2, 3$. The topological magnetic current, which is also the topological current density of the system is[25]

$$k_\mu = \frac{1}{8\pi} \epsilon_{\mu\nu\rho\sigma} \epsilon_{abc} \partial^\nu \hat{\Phi}^a \partial^\rho \hat{\Phi}^b \, \partial^\sigma \hat{\Phi}^c \,. \tag{7}$$

Therefore the corresponding conserved topological magnetic charge is

$$M = \int d^3x \, k_0 = \frac{1}{4\pi} \oint d^2\sigma_i \, B_i \,. \tag{8}$$

In the BPS limit when the Higgs potential vanishes, the energy can be written in the form

$$\begin{aligned} E &= \mp \int \partial_i (B_i^a \Phi^a) d^3x + \int \frac{1}{2} (B_i^a \pm D_i \Phi^a)^2 \, d^3x \\ &= \mp \int \partial_i (B_i^a \Phi^a) d^3x = 4\pi M \frac{\mu}{\sqrt{\lambda}} \,, \end{aligned} \tag{9}$$

where $M$ is the "topological charge" when the vacuum expectation value of the Higgs field, $\frac{\mu}{\sqrt{\lambda}}$, is nonzero coupled with some nontrivial topological structure of the fields at large $r$.

## 3. The Dyons

### 3.1. *The ansatz*

The static gauge fields and Higgs field that will lead to the dyon solutions of one-half monopole charge are given respectively by,[18–20,24]

$$
\begin{aligned}
A_0^a &= \sinh\gamma(\Phi_1 \hat{r}^a + \Phi_2 \hat{\theta}^a)\,,\\
A_i^a &= \frac{1}{r}\psi(r)(\hat{\theta}^a\hat{\phi}_i - \hat{\phi}^a\hat{\theta}_i) + \frac{1}{r}R(\theta)(\hat{\phi}^a\hat{r}_i - \hat{r}^a\hat{\phi}_i)\,,\\
\Phi^a &= \cosh\gamma(\Phi_1 \hat{r}^a + \Phi_2 \hat{\theta}^a)\,,
\end{aligned}
\tag{10}
$$

where $\Phi_1 = \frac{1}{r}\psi(r)$, $\Phi_2 = \frac{1}{r}R(\theta)$ and $\gamma$ is an arbitrary constant. The spherical coordinate orthonormal unit vectors, $\hat{r}^a$, $\hat{\theta}^a$, and $\hat{\phi}^a$ are defined by

$$
\begin{aligned}
\hat{r}^a &= \sin\theta\cos\phi\delta_1^a + \sin\theta\sin\phi\delta_2^a + \cos\theta\delta_3^a\,,\\
\hat{\theta}^a &= \cos\theta\cos\phi\delta_1^a + \cos\theta\sin\phi\delta_2^a - \sin\theta\delta_3^a\,,\\
\hat{\phi}^a &= -\sin\phi\delta_1^a + \cos\phi\delta_2^a\,,
\end{aligned}
\tag{11}
$$

where $r = \sqrt{x^i x_i}$, $\theta = \cos^{-1}(x_3/r)$, and $\phi = \tan^{-1}(x_2/x_1)$. Ansatz (10) satisfies the Coulomb gauge, $\partial^i A_i^a = 0$.

### 3.2. *The solutions*

Upon substituting ansatz (10) into the equations of motion (4), the resulting equations are simplified to two first order differential equations,

$$
\begin{aligned}
r\psi' + \psi - \psi^2 &= -m(m+1)\,,\\
\dot{R} + R\cot\theta - R^2 &= m(m+1)\,,
\end{aligned}
\tag{12}
$$

where $\psi'$ means $\frac{\partial\psi}{\partial r}$ and $\dot{R}$ means $\frac{\partial R}{\partial\theta}$. The solutions for $\psi$ and $R$ are then given respectively by

$$
\begin{aligned}
\psi(r) &= \frac{(m+1) - m(br)^{2m+1}}{1 + (br)^{2m+1}}\,,\\
R(\theta) &= (m+1)\csc\theta\left\{\cos\theta - \frac{(P_{m+1}(\cos\theta) + aQ_{m+1}(\cos\theta))}{(P_m(\cos\theta) + aQ_m(\cos\theta))}\right\},
\end{aligned}
\tag{13}
$$

where $P_m$ and $Q_m$ are the Legendre polynomials of the first and second kind of degree $m$, respectively, and $-1/2 \leq m < 0$. For solutions regular along the positive $z$-axis, we required, $R(0) = 0$, and the integration constant $a$ is set to zero. However $R(\pi) = \infty$. It is this boundary condition that gives rise to the string singularity in the gauge field and the magnetic field. The integration constant $b$ is just a scaling factor and without any loss in generality, we set $b = 1$ and the boundary conditions for $\psi(r)$ are $\psi(0) = m + 1$ and $\psi(\infty) = -m$. In these dyon solutions there are no zeros of the Higgs field and the Higgs unit vector is only indeterminate at the origin where a monopole of half unit charge is located.

From the ansatz (10), the 't Hooft gauge potential becomes,

$$A_\mu = \hat{\Phi}^a A^a_\mu = \frac{\sinh\gamma}{r}\sqrt{\psi(r)^2 + R(\theta)^2}\delta^0_\mu\,. \tag{14}$$

Hence the 't Hooft electric field is nonzero when $\gamma \neq 0$ and the monopole becomes a dyon. The 't Hooft magnetic field is independent of the gauge potentials $A^a_\mu$.

### 3.3. *The 't Hooft magnetic field*

To calculate for the 't Hooft magnetic field $B_i$, which is independent of the gauge fields $A^a_\mu$, we rewrite the Higgs field from the spherical to the Cartesian coordinate system,[18–20]

$$\begin{aligned}\Phi^a &= \Phi_1\hat{r}^a + \Phi_2\hat{\theta}^a + \Phi_3\hat{\phi}^a\\ &= \tilde{\Phi}_1\delta^{a1} + \tilde{\Phi}_2\delta^{a2} + \tilde{\Phi}_3\delta^{a3}\end{aligned} \tag{15}$$

where

$$\begin{aligned}\tilde{\Phi}_1 &= \sin\theta\cos\phi\Phi_1 + \cos\theta\cos\phi\Phi_2 - \sin\phi\Phi_3 = |\Phi|\cos\alpha\sin\beta\,,\\ \tilde{\Phi}_2 &= \sin\theta\sin\phi\Phi_1 + \cos\theta\sin\phi\Phi_2 + \cos\phi\Phi_3 = |\Phi|\cos\alpha\cos\beta\,,\\ \tilde{\Phi}_3 &= \cos\theta\Phi_1 - \sin\theta\Phi_2 = |\Phi|\sin\alpha\,.\end{aligned} \tag{16}$$

The Higgs unit vector can be simplified to

$$\hat{\Phi}^a = \cos\alpha\sin\beta\delta^{a1} + \cos\alpha\cos\beta\delta^{a2} + \sin\alpha\delta^{a3}\,, \tag{17}$$

where

$$\sin\alpha = \frac{\psi\cos\theta - R\sin\theta}{\sqrt{\psi^2 + R^2}}\,, \qquad \beta = \frac{\pi}{2} - \phi\,,$$

and the 't Hooft magnetic field is reduce to only the $\hat{r}_i$ and $\hat{\theta}_i$ components,

$$B_i = -\frac{1}{r^2\sin\theta}\left\{\frac{\partial\sin\alpha}{\partial\theta}\right\}\hat{r}_i + \frac{1}{r\sin\theta}\left\{\frac{\partial\sin\alpha}{\partial r}\right\}\hat{\theta}_i\,. \tag{18}$$

The function, $\sin\alpha$, is a bounded function over all space and the 't Hooft magnetic field is regular everywhere except at the location of the monopole at the origin and along the negative $z$-axis where there is a semi infinite string singularity. When $m = -1/2$ and $m = 0$, $\sin\alpha$ is independent of $r$ and the magnetic field is purely radial in direction. We also notice that, $B_i$, can be written as

$$B_i = \epsilon_{ijk}\partial^j(\sin\alpha)\partial^k\beta = \epsilon_{ijk}\partial^j(\sin\alpha\partial^k\beta)\,, \tag{19}$$

and that a suitable Maxwell four-vector gauge potential for this magnetic field is

$$\mathcal{A}_0 = \frac{\sinh\gamma}{r}\sqrt{\psi^2 + R^2}\,, \qquad \mathcal{A}_i = (\sin\alpha - 1)\partial_i\beta = -\frac{(\sin\alpha - 1)}{r\sin\theta}\hat{\phi}_i\,. \tag{20}$$

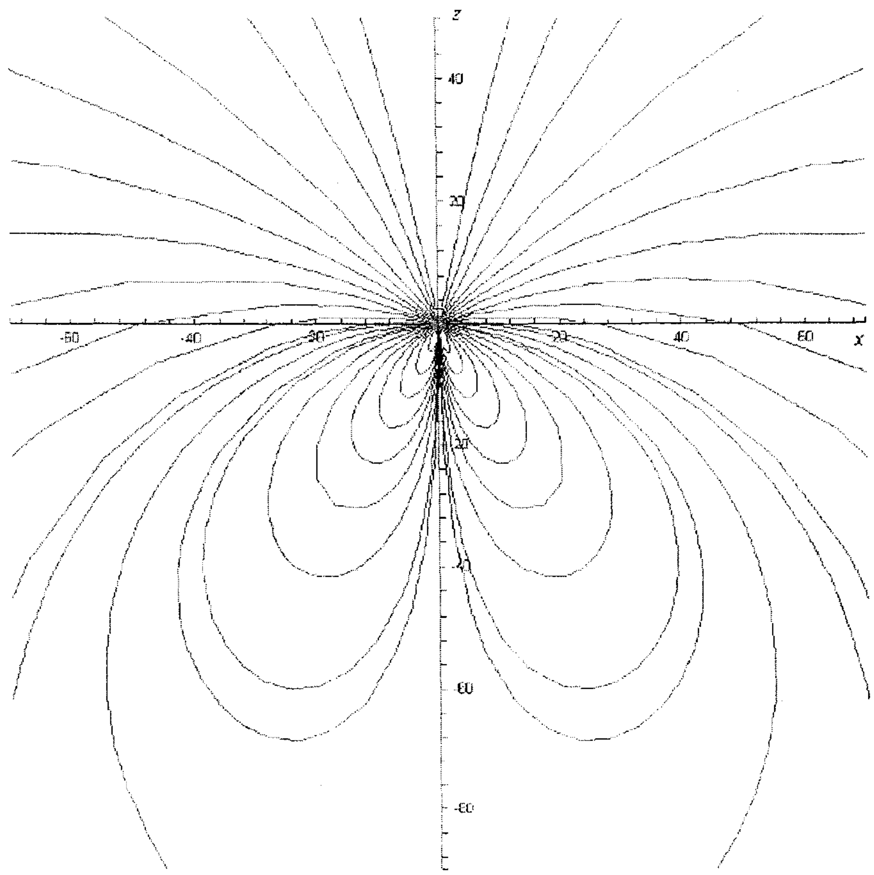


Fig. 1. The magnetic field lines of the $m = -0.01$ dyon. The one-half monopole is located at $r = 0$.

When $m = 0$, the gauge potential, $\mathcal{A}_i$, is just the usual Dirac string potential and it is singular along the negative $z$-axis. However when $-1/2 \leq m < 0$, the gauge potential $\mathcal{A}_i$ still possesses a similar string singularity along the negative $z$-axis, but it is no longer the Dirac string.

From Eq. (19), it is obvious that the magnetic field is always perpendicular to the gradients of $\sin\alpha$ and $\beta$. Hence the magnetic field lines lie on the line $\sin\alpha = k$, where $-1 < k < 1$, and $\phi =$ constant. By plotting $\sin\alpha = k$ on a vertical plane through the origin; we manage to draw the magnetic field lines for the configurations when $m = -0.01$, Fig. 1; $m = -0.05$, Fig. 2; and $m = -0.2$, Fig. 3. We have notice that for every field line through the origin, one end of the line goes to infinity, whereas the other end will make a loop and return back to $r = 0$. Therefore the magnetic flux going to infinity is only half of that of a monopole of unit charge.

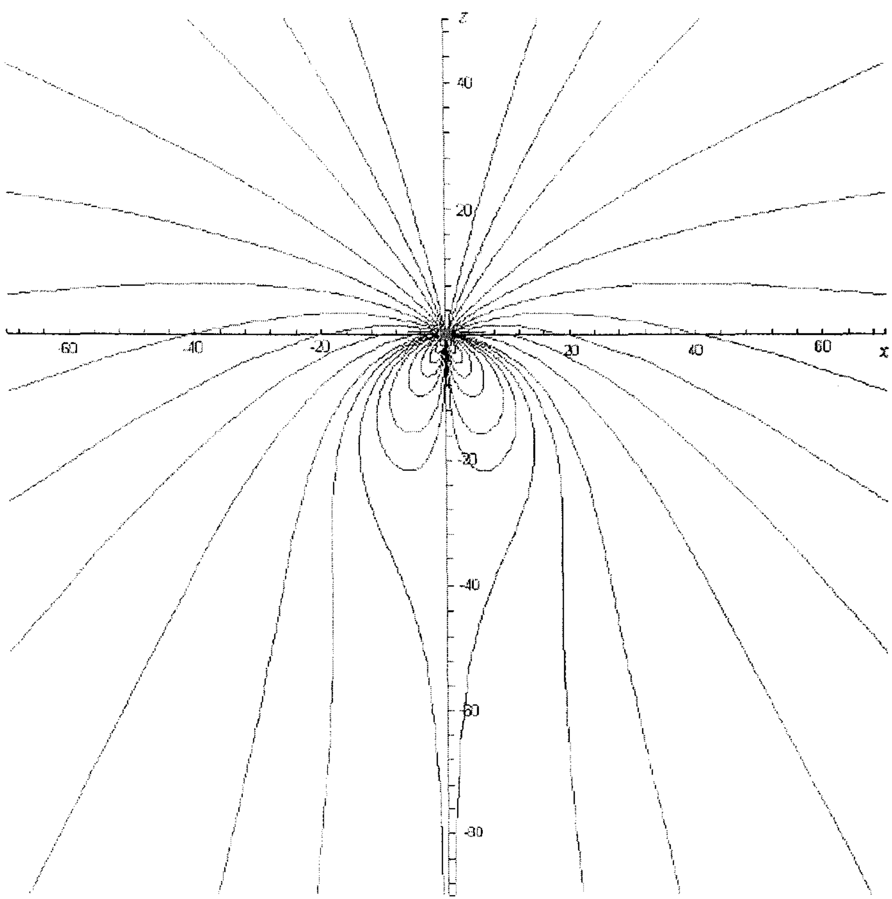


Fig. 2. The magnetic field lines of the $m = -0.05$ dyon. The one-half monopole is located at $r = 0$.

### 3.4. *The monopole charge*

From Eq. (8), the net magnetic charge enclosed by the sphere at infinity is calculated to be,

$$\begin{aligned} M_\infty = -\frac{1}{2}\sin\alpha\Big|_{0,r\to\infty}^{\pi} &= \frac{1}{2} \quad \text{when} \quad -\frac{1}{2} \le m < 0\,, \\ &= 1 \quad \text{when} \quad m = 0\,. \end{aligned} \tag{21}$$

Hence the magnetic charge of the system is always one-half over all space when $m$ is a noninteger less than zero and it is concentrated at only one point in space at $r = 0$. When $m = 0$, the half monopole becomes a unit monopole.

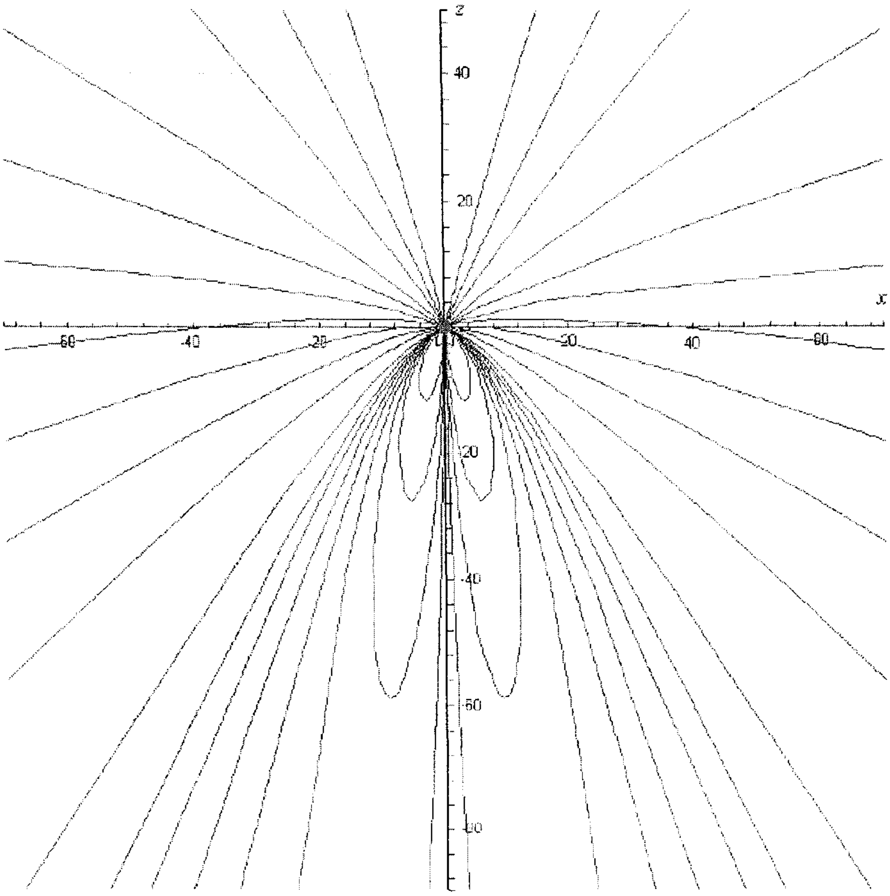


Fig. 3. The magnetic field lines of the $m = -0.2$ dyon. The one-half monopole is located at $r = 0$.

1 ### 3.5. *The 't Hooft electric field*

The 't Hooft electric field of the dyons is given by

$$E_i = -\partial_i A_0 = \sinh\gamma\left\{\sqrt{\psi^2 + R^2} - \frac{\psi(\psi + m)(\psi - m - 1)}{\sqrt{\psi^2 + R^2}}\right\}\left(\frac{\hat{r}_i}{r^2}\right)$$
$$- \sinh\gamma\left\{\frac{R(R^2 - R\cot\theta + m(m + 1))}{\sqrt{\psi^2 + R^2}}\right\}\left(\frac{\hat{\theta}_i}{r^2}\right). \qquad (22)$$

Unlike the magnetic field, the electric field depends on the constant $\gamma$. Hence the electric field can be switched off by setting $\gamma = 0$. As the parameter $m$ increases

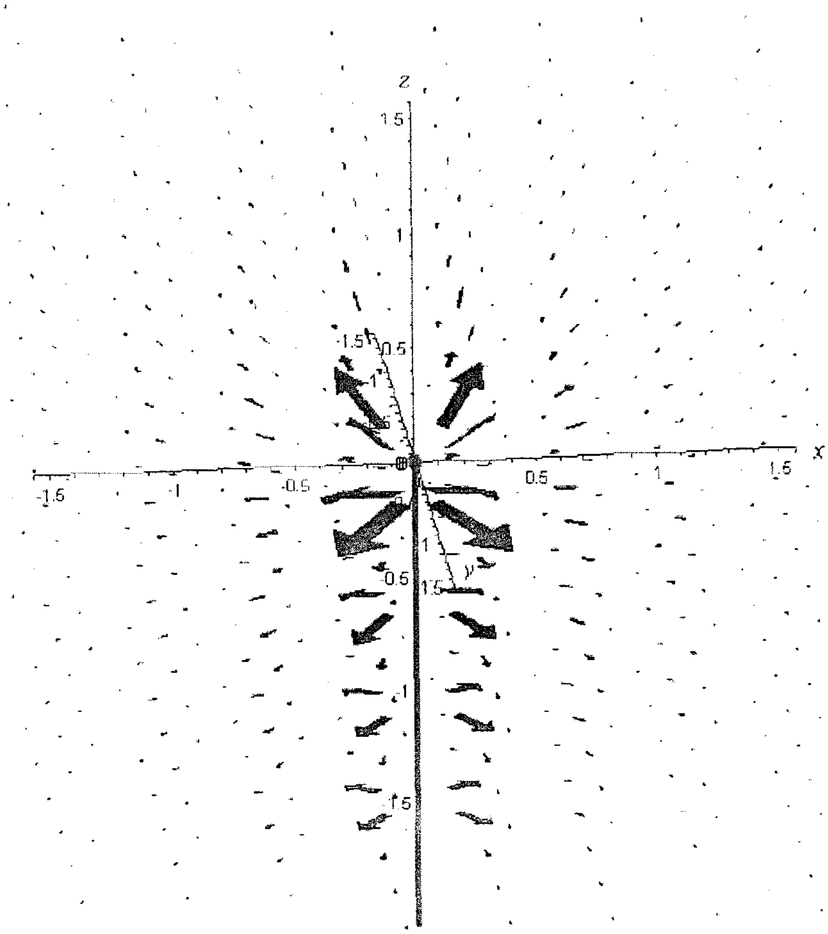


Fig. 4. The electric field of the $m = -0.5$ dyon. The positive electric charges are concentrated at $r = 0$ and along the negative $z$-axis.

from $-1/2$ to zero, the $\hat{\theta}_i$ component of the electric field diminishes to zero while the radial component approaches

$$E_i = \left\{ \frac{2}{(1+r)} - \frac{1}{(1+r)^2} \right\} \frac{\hat{r}_i}{r^2}, \tag{23}$$

and the solution becomes spherically symmetric. The 3D field plot of the electric field for different values of $m$ in the range $-1/2 \leq m < 0$, shows the presence of a positive point charge at $r = 0$ and a positive line charge along the negative $z$-axis. Figure 4 shows the 3D field plot for the case of $m = -1/2$. The electric field which is pointing radially outward diminishes as $1/r^2$ along the negative $z$-axis.

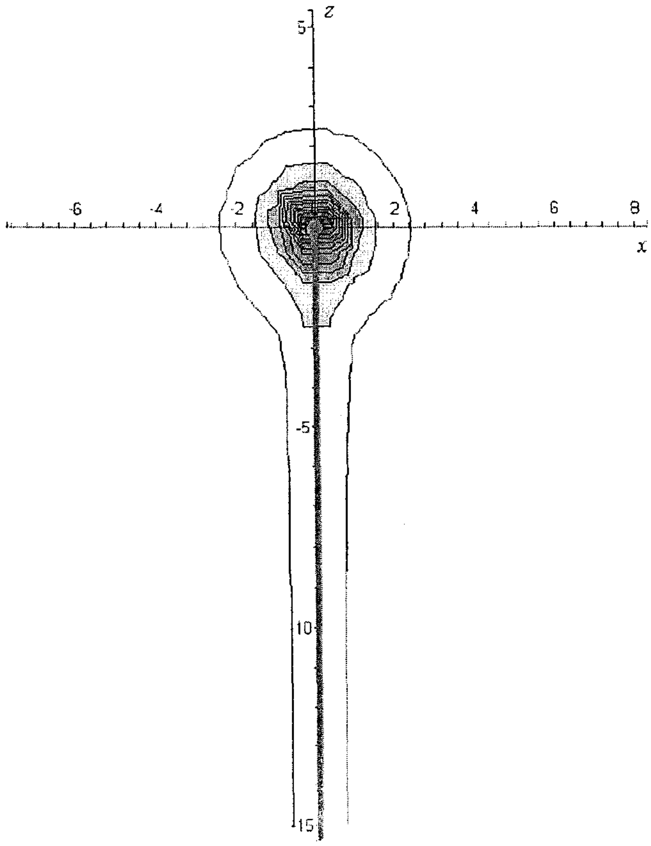


Fig. 5. The contour plot of the electric charge density distribution of the $m = -0.05$ dyon. The negative electric charge density distribution is in blue and the positive charges are indicated in red.

### 3.6. *The electric charge*

The electric charge densities, $q = \partial^i E_i$, of these dyon solutions consist of a negative cloud charge distribution concentrated in regions around the origin and along the negative $z$-axis diminishing along the axis as $\frac{1}{r^3}$. The positive charge densities are delta functions distributions at $r = 0$ and along the negative $z$-axis. Figures 5 and 6 show the electric charge density distribution for the $m = -0.05$ and $m = -0.5$ dyons respectively. The negative electric cloud charge density distribution is in blue and the positive charges are indicated in red. The charge distribution along the negative $z$-axis is greatest when $m = -1/2$ and continuously diminishes to zero as $m$ approaches zero when the dyon's monopole charge changes from half to one. The total electric charge of the dyons can be obtained by Gauss' law,

$$Q = \int_{r\to\infty} E_i \hat{r}_i r^2 \sin\theta \, d\theta \, d\phi = 2\pi \sinh\gamma \int \sqrt{m^2 + R(\theta)^2} \sin\theta \, d\theta \,. \qquad (24)$$

singularity along the negative $z$-axis. Configurations of one-half monopole charge that possess axial symmetry and a semi infinite string singularity both in the gauge potentials and the electromagnetic fields have also been discussed in the literature.[23]

Calculations show that the electric charges that give rise to the $\hat{\theta}_i$ component of the electric field are composed of a negative cloud charge distribution around the negative $z$-axis and a positive delta function charge distribution along the negative $z$-axis which exactly canceled out the charges of each other at $r$ infinity.

The electric charges that give rise to the radial component of the electric field are all concentrated at $r = 0$ when $m = -1/2$. Hence $Q = Q_0$ for the $m = -1/2$ dyon. However when, $-1/2 < m \leq 0$, there is a negative cloud charge distribution around the origin and this negative cloud charge increases in intensity until it reaches its maximum value at $m = 0$ where it totally screened off the positive delta function source at $r = 0$ giving a zero net electric charge for the Wu-Yang type dyon when $m = 0$. Hence for the Wu-Yang type dyon, $Q = 0$, and $Q_0 = 4\pi \sinh\gamma$.

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H(i) 5.

# A Numerical Investigation on the MAP Solutions of the SU(2) YMH Theory


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## Abstract

We notice that it is not necessary to describe the monopole-antimonopole pair (MAP) and monopole-antimonopole chains (MAC) solutions in terms of $\theta$-winding number $m$ greater than one as they can also be parameterized by a single integer $s$. Here we study the MAP solution of the SU(2) Yang-Mills-Higgs theory which belongs to the topological trivial sector. This solution is parameterized by $\theta$-winding number one, $\phi$-winding number one and integer $s = 1$.


## I. INTRODUCTION

The SU(2) Yang-Mills-Higgs (YMH) field theory, with the Higgs field in the adjoint representation, are known to possess both the magnetic monopole [1,2] and multimonopole solutions [3]. Monopole solutions with unit magnetic charge and finite energy are spherically symmetric [1,2]. Multimonopole with finite energy cannot be spherically symmetric [4] and possess at most axial symmetry [3].

In the limit of vanishing Higgs potential, the Bogomol'nyi-Prasad-Sommerfield (BPS) limit, exact monopole and multimonopole solutions [2,3] are known. These BPS solutions possess finite and minimal energies. However, when the Higgs potential is finite, only numerical solutions [1] are known.

Recently Kleihaus et al. constructed non-Bogomol'nyi BPS solutions which satisfy only the second order field equations but not the first order Bogomol'nyi equations. Their solutions possess only axial symmetry and correspond to a monopole-antimonopole pair (MAP) [5], and monopole-antimonopole chain (MAC) [6]. These MAP and MAC solutions are parameterized by $\theta$-winding number $m$ ($> 1$) and $\phi$-winding number $n = 1$.

In Ref. [7], we show that the $\theta$-winding number of the MAP and MAC solutions can be reduced to one and a single integer parameter $s$. In other words, there exist an equivalent form of the solutions with normal $\theta$-winding number $m = 1$ and integer parameter $s$ for all the solutions of $\theta$-winding number $m$.

In this paper we compute the numerical MAP solutions with $m = 1, s = 1$ and $m = 2, s = 0$. Both the asymptotic conditions $s = 0$ and $s = 1$ correspond to a pure gauge at spatial infinity and hence the system possess zero net magnetic charge. We connect the asymptotic solutions with the trivial vacuum at the origin numerically by using mathematical softwares Maple and Matlab.

We briefly review the SU(2) YMH field theory in the next section. We present the magnetic ansatz and some of its basic properties in section III. In section IV, we discussed the net magnetic charge of the configuration and we present numerical results in section V. We conclude our results and give some comments in section VI.

## II. THE SU(2) YANG-MILLS-HIGGS-THEORY

The SU(2) YMH theory admits the triplet gauge field $A^a_\mu$ which are the Yang-Mills vector fields coupled to a scalar Higgs triplets field $\Phi^a$ in 3+1 dimensions. The index $a$ is the SU(2) internal space index and for a given $a$, $\Phi^a$ is a scalar whereas $A^a_\mu$ is a vector under Lorentz transformation. The Lagrangian in 3+1 dimensions is given by

$$L = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}D^\mu\Phi^a D_\mu\Phi^a - \frac{1}{4}\lambda\left(\Phi^a\Phi^a - \frac{\mu^2}{\lambda}\right)^2, \tag{1}$$

where the Higgs field mass, $\mu$, and the strength of the Higgs potential, $\lambda$, are constants. The vacuum expectation value of the Higgs field is then given by $\mu/\sqrt{\lambda}$ . The Lagrangian (1) is gauge invariant under the set of independent local SU(2) transformations at each space time point. The covariant derivative of the Higgs field and the gauge field strength tensor are given respectively by

$$D_\mu\Phi^a = \partial_\mu\Phi^a + \epsilon^{abc} A^b_\mu\Phi^c, \text{ and } F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu. \tag{2}$$

Since the gauge field coupling constant $g$ can be scaled away, we can set $g$ to one without any loss of generality. The metric used is $g_{\mu\nu} = (-+++)$. The SU(2) internal group indices $a, b, c$ run from 1 to 3 and the spatial indices are $\mu, \nu, \alpha = 0, 1, 2,$ and 3 in Minkowski space.

The equations of motion that follow from the Lagrangian (1) are

$$D^\mu F^a_{\mu\nu} = \partial^\mu F^a_{\mu\nu} + \epsilon^{abc} A^{b\mu} F^c_{\mu\nu} = \epsilon^{abc} \Phi^b D_\nu\Phi^c, \; D^\mu D_\mu\Phi^a = -\lambda\Phi^a\left(\Phi^b\Phi^b - \frac{\mu^2}{\lambda}\right). \tag{3}$$

The 't Hooft electromagnetic field tensor as proposed by 't Hooft [1], is given by

$$F_{\mu\nu} = \hat{\Phi}^a F^a_{\mu\nu} - \epsilon^{abc} \hat{\Phi}^a D_\mu\hat{\Phi}^b D_\nu\hat{\Phi}^c = \partial_\mu A_\nu - \partial_\nu A_\mu - \epsilon^{abc} \hat{\Phi}^a\partial_\mu\hat{\Phi}^b\partial_\nu\hat{\Phi}^c, \tag{4}$$

where $A_\mu = \hat{\Phi}^a A_\mu^a$, the Higgs field unit vector $\hat{\Phi}^a = \Phi^a / |\Phi|$ and the Higgs field magnitude $|\Phi| = \sqrt{\Phi^a \Phi^a}$. The Abelian electric field is $E_i = F_{0i}$, and the Abelian magnetic field is $B_i = -\frac{1}{2} \epsilon_{ijk} F_{jk}$. The topological magnetic current [8] which is also the topological current density of the system is defined as

$$k_\mu = \frac{1}{8\pi} \epsilon_{\mu\nu\rho\sigma} \epsilon_{abc} \partial^\nu \hat{\Phi}^a \partial^\rho \hat{\Phi}^b \partial^\sigma \hat{\Phi}^c, \tag{5}$$

and the corresponding conserved topological magnetic charge is

$$M = \int d^3x k_0 = \frac{1}{8\pi} \int \epsilon_{ijk} \epsilon^{abc} \partial_i \left( \hat{\Phi}^a \partial_j \hat{\Phi}^b \partial_k \hat{\Phi}^c \right) d^3x = \frac{1}{8\pi} \oint d^2\sigma_i \left( \epsilon_{ijk} \epsilon^{abc} \hat{\Phi}^a \partial_j \hat{\Phi}^b \partial_k \hat{\Phi}^c \right)$$

$$= \frac{1}{4\pi} \oint d^2\sigma_i B_i. \tag{6}$$

Our work is restricted to the static case where $A_0^a = 0$. Hence the conserved energy of the system for the static case reduces to

$$E = \int d^3x \left( \frac{1}{2} B_i^a B_i^a + \frac{1}{2} D_i \Phi^a D_i \Phi^a + \frac{1}{4} \lambda \left( \Phi^a \Phi^a - \frac{\mu^2}{\lambda} \right)^2 \right). \tag{7}$$

Here *i, j, k* which are the three space indices run from 1, 2, and 3. This energy vanishes when the gauge potential, $A_i^a$ is zero or when $A_i^a$ is a pure gauge, and when $\Phi^a \Phi_a = \mu^2 / \lambda$ and $D_i \Phi^a = 0$. In this paper, we consider the case with vanishing Higgs potential, $\lambda = 0$.

## III. THE MAGNETIC ANSATZ

We make use of the static magnetic ansatz [5,6] to solve for the monopoles solutions here. The gauge fields and the Higgs field are given respectively by

$$A_i^a = \frac{1}{r}(-\psi_1\ \hat{\phi}^a \hat{\theta}_i + \psi_2\ \hat{\theta}^a \hat{\phi}_i) + \frac{1}{r}(R_1\ \hat{\phi}^a \hat{r}_i - R_2\ \hat{r}^a \hat{\phi}_i),\ \ \Phi^a = \Phi_1 \hat{r}^a + \Phi_2 \hat{\theta}^a, \tag{8}$$

where $\psi_1$, $\psi_2$, $R_1$, $R_2$, $\Phi_1$ and $\Phi_2$ are profile functions of $r$ and $\theta$. The spherical coordinate orthonormal unit vectors, $\hat{r}_i$, $\hat{\theta}_i$, and $\hat{\phi}_i$ are defined by

$$\hat{r}_i = \sin\theta \cos\phi\ \ \delta_{1i} + \sin\theta \sin\phi\ \ \delta_{2i} + \cos\theta\ \ \delta_{3i},$$

$$\hat{\theta}_i = \cos\theta\cos\phi \;\; \delta_{1i} + \cos\theta\sin\phi \;\; \delta_{2i} - \sin\theta \;\; \delta_{3i},$$
$$\hat{\phi}_i = -\sin\phi \;\; \delta_{1i} + \cos\phi \;\; \delta_{2i}, \tag{9}$$

and the isospin coordinate orthonormal unit vectors, $\hat{r}^a$, $\hat{\theta}^a$, and $\hat{\phi}^a$ are defined by

$$\hat{r}^a = \sin m\theta\cos n\phi \;\; \delta_1^a + \sin m\theta\sin n\phi \;\; \delta_2^a + \cos m\theta \;\; \delta_3^a,$$
$$\hat{\theta}^a = \cos m\theta\cos n\phi \;\; \delta_1^a + \cos m\theta\sin n\phi \;\; \delta_2^a - \sin m\theta \;\; \delta_3^a,$$
$$\hat{\phi}^a = -\sin n\phi \;\; \delta_1^a + \cos n\phi \;\; \delta_2^a, \tag{10}$$

where $r = \sqrt{x^i x_i}$, $\theta = \cos^{-1}(x_3/r)$, and $\phi = \tan^{-1}(x_2/x_1)$.

With ansatz (8) the field equations (3) reduce to six PDEs in $r$ and $\theta$. The ansatz possesses a residual U(1) gauge symmetry. To fix the gauge, we impose the gauge condition $r\partial_r R_1 - \partial_\theta \psi_1 = 0$.

In Refs. [5,6], Kleihaus et. al. consider MAP and MAC solutions which is parameterized by the $\theta$-winding number $m$ (>1) and $\phi$-winding number $n$ (=1). In Ref. [7], we show that there always exist an equivalent form of the solutions with normal winding number $m = 1$ and an integer $s$ for all the solutions with $\theta$ winding number.

The MAP solution possesses exact asymptotic solutions at both small and large $r$. Upon reducing these solutions to the $m = 1$ form with integer $s$, both the MAP and MAC solutions correspond to the trivial vacuum ($s = 0$) at small $r$. However, at large $r$, the MAP solutions tend to a different sector of the vacuum with parameter $s = 1, 2, 3$... Hence the MAP solutions correspond to a one monopole-antimonopoe pair when $s = 1$, a two monopole-antimonopole pair when $s = 2$, and so on with net topological magnetic charge zero. Here we compute numerically the MAP solution with $m = 1$, $s = 1$, $n = 1$ and $m = 2$, $s = 0$, $n = 1$.

The boundary conditions at the origin are

$$\psi_1(0,\theta) \to 0,\; \psi_2(0,\theta) \to 0,\; R_1(0,\theta) \to 0,\; R_2(0,\theta) \to 0,$$
$$\Phi_1(0,\theta) \to -\xi\cos\theta,\; \Phi_2(0,\theta) \to \xi\sin\theta, \tag{11}$$

where $\xi$ is arbitrary constant and the boundary conditions at $r$ infinity are

$$\psi_1(\infty,\theta) \to 2,\; \psi_2(\infty,\theta) \to 2n,\; R_1(\infty,\theta) \to 0,\; R_2(\infty,\theta) \to 0,$$
$$\Phi_1(\infty,\theta) \to \cos\theta,\; \Phi_2(\infty,\theta) \to \sin\theta, \tag{12}$$

Regularity on the $z$-axis (at $\theta = 0$ and $\theta = \pi$) requires

$$R_1 = 0,\ R_2 = 0,\ \Phi_2 = 0,\ \partial_\theta \psi_1 = 0,\ \partial_\theta \psi_2 = 0,\ \partial_\theta \Phi_1 = 0. \tag{13}$$

The equations of motion (3) are then solved numerically by using ansatz (8) with the boundary conditions (11)-(13). The constant $\xi$ will be determined by the results of the numerical calculations.

## IV. THE NET MAGNETIC CHARGE

To calculate for the Abelian magnetic field $B_i$, we rewrite the Higgs field in Eq. (8) from the spherical to the Cartesian coordinate system, [5,6]

$$\Phi^a = \Phi_1 \hat{r}^a + \Phi_2 \hat{\theta}^a + \Phi_3 \hat{\phi}^a = \tilde{\Phi}_1 \delta^{a1} + \tilde{\Phi}_2 \delta^{a2} + \tilde{\Phi}_3 \delta^{a3}, \tag{14}$$

where
$$\begin{aligned}
\tilde{\Phi}_1 &= \sin m\theta \cos n\phi\ \Phi_1 + \cos m\theta \cos n\phi\ \Phi_2 - \sin n\phi\ \Phi_3 = |\Phi| \cos\alpha \sin\beta, \\
\tilde{\Phi}_2 &= \sin m\theta \sin n\phi\ \Phi_1 + \cos m\theta \sin n\phi\ \Phi_2 + \cos n\phi\ \Phi_3 = |\Phi| \cos\alpha \cos\beta, \\
\tilde{\Phi}_3 &= \cos m\theta\ \Phi_1 - \sin m\theta\ \Phi_2 = |\Phi| \sin\alpha.
\end{aligned} \tag{15}$$

The Higgs unit vector is then simplified to

$$\hat{\Phi}^a = \cos\alpha \sin\beta\ \ \delta^{a1} + \cos\alpha \cos\beta\ \ \delta^{a2} + \sin\alpha\ \ \delta^{a3}, \tag{16}$$

$$\text{where } \sin\alpha = \frac{\Phi_1 \cos m\theta - \Phi_2 \sin m\theta}{\sqrt{\Phi_1^2 + \Phi_2^2}},\quad \beta = \gamma - \phi,\quad \gamma = \pi/2, \tag{17}$$

and the Abelian magnetic field is found to be

$$B_i = -\frac{1}{r^2 \sin\theta}\left(\frac{\partial \sin\alpha}{\partial \theta}\frac{\partial \beta}{\partial \phi} - \frac{\partial \sin\alpha}{\partial \phi}\frac{\partial \beta}{\partial \theta}\right)\hat{r}_i + \frac{1}{r\sin\theta}\left(\frac{\partial \sin\alpha}{\partial \phi}\frac{\partial \beta}{\partial r} - \frac{\partial \sin\alpha}{\partial r}\frac{\partial \beta}{\partial \phi}\right)\hat{\theta}_i. \tag{18}$$

The magnetic charge is defined as

$$4\pi M = \oint d^2\sigma_i B_i = \int B_i (r^2 \sin\theta\ \ d\theta) \hat{r}_i\ \ d\phi. \tag{19}$$

Here we define the magnetic charge enclosed by the upper hemisphere of infinite radius as $M_+$ whereas the magnetic charge enclosed by the lower hemisphere of infinite radius is denoted by $M_-$. The value of $M_+$ is calculated to be

$$M_+ = -\frac{1}{2}\sin\alpha\Big|_{0,\, r\to\infty}^{\pi/2} = +1\ . \tag{20}$$

The upper hemisphere then possesses a positive charged monopole. Similarly, considering the lower hemisphere by integrating Eq. (19) from $\theta = \pi/2$ to $\theta = \pi$ gives

$$M_{-} = -\frac{1}{2}\sin\alpha\Big|_{\pi/2,\, r\to\infty}^{\pi} = -1, \tag{21}$$

which correspond to a negative charged antimonopole.

These calculations show indeed that the configuration possesses a monopole-antimonopole pair, with the monopole situated on the positive $z$-axis and the antimonopole at equidistance on the negative $z$-axis. At $r$ infinity for surface enclosing both charges, their contributions compensate and yields zero net magnetic charge.

## V. THE NUMERICAL RESULTS

The asymptotic conditions (12) correspond to a pure gauge vacuum at spatial infinity. We connect this asymptotic condition with the trivial vacuum at the origin (11) numerically by using mathematical softwares Maple and Matlab.

The equations of motion are discretized on a non-equidistant grid, covering the integration region $0 \le X \le 1$ and $0 \le \theta \le \pi$, where $X$ is the finite interval compactified coordinate. The $X$ and $\theta$ grid are subdivided into $M$ and $N$ divisions. The best accuracy our computer is able to support is $M = 40$ and $N = 25$.

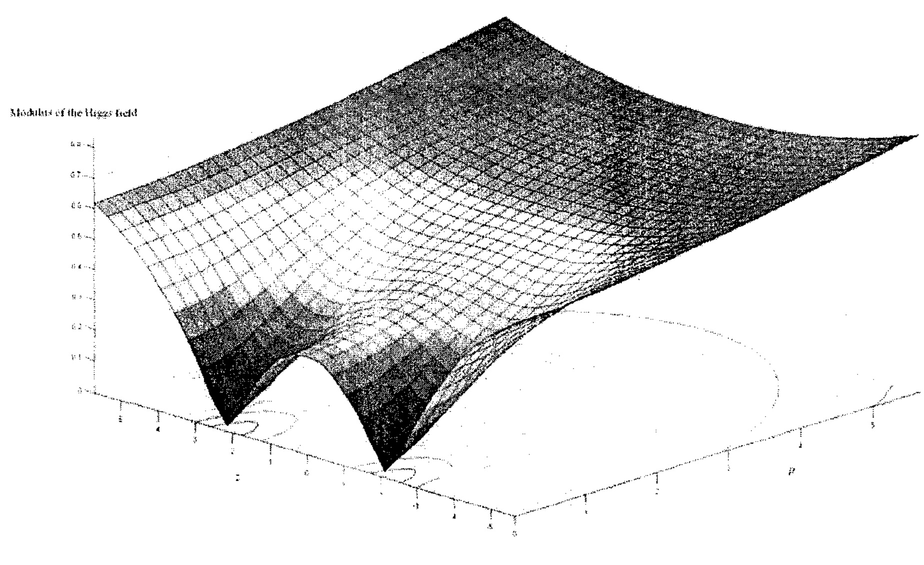


Fig.(1) The modulus of the Higgs field as a function of $\rho$ and $z$.

The numerical method used is the Gauss-Newton method and it is a good iterative method to obtain numerically accurate solutions. First the set of six PDEs are transformed into a system of non-linear equations by considering second order finite difference approximation. After providing good initial guess to the system of non-linear equations, the solver will iterate and converge to the true numerical answers.

Our result confirms that the boundary conditions (11)-(13) corresponds to monopole-antimonopole pair solution. In Fig. (1) we plot the modulus of the Higgs field as a functions of $\rho$ and $z$ where $\rho = \sqrt{x^2 + y^2}$ . The magnetic poles are located at the points where the modulus of Higgs field is zero. In particular, when $M = 40$, $N = 25$, the location where the magnetic poles are situated is $z_0 = 2.4$. We also notice that with increasing accuracy in the $\theta$ grid from $N = 15$ to $N = 25$, the two MAP solutions ($m = 1$, $s = 1$ and $m = 2$, $s = 0$) appear to approach each other from different direction, Fig. (2). Hence it is interesting to know whether the two will coincide when the accuracy in $r$ and $\theta$ grid is high enough. This will be done with more powerful computer in a later work.

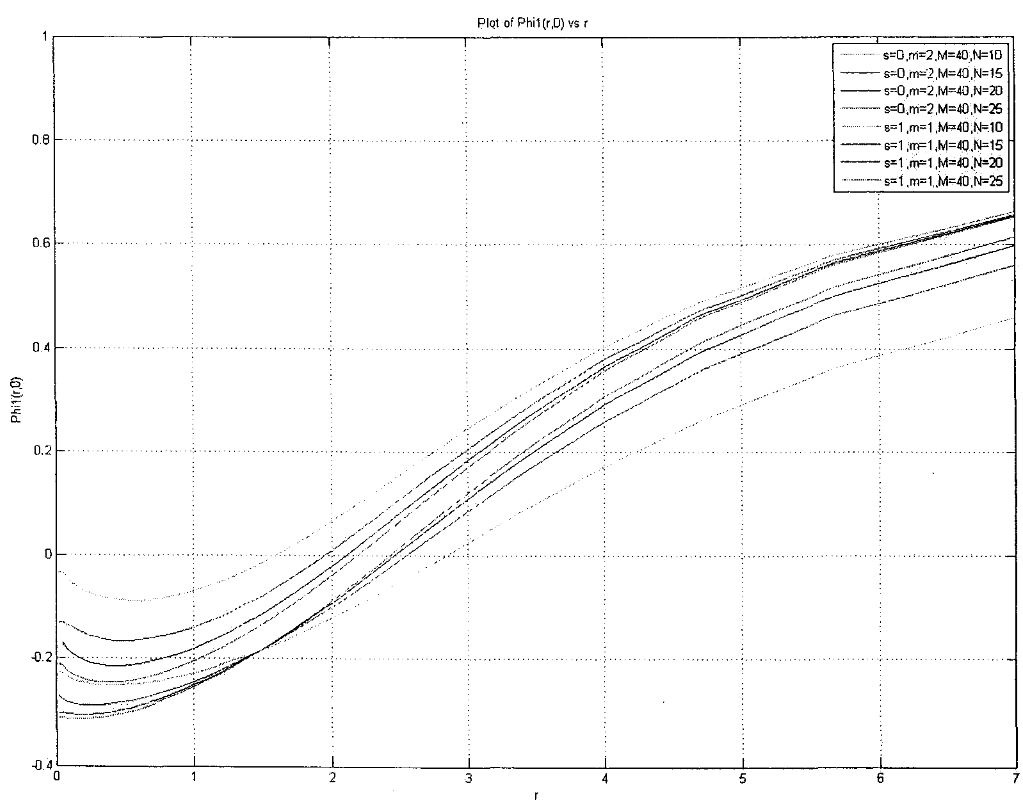


Fig.(2) A plot of $\Phi(r,0)$ versus $r$ for $M = 40$, $N = 10, 15, 20, 25$ for the MAP solutions parameterized by $m = 2$, $s = 0$, $n = 1$ (the first four graph on the left) and $m = 1$, $s = 1$, $n = 1$ (the next four graphs on the right).

## VI. CONCLUSION

Following results in Ref. [7] where it is possible to alternatively describe MAP and MAC solutions in terms of integer $s$, and winding number $m = 1$, we study such axially symmetric monopole-antimonopole pair solutions, parameterized by integer $m = 1$, $s = 1$ and $n = 1$. This solution resides in the topological vacuum sector and it represents a monopole on the positive $z$-axis and antimonopole on the negative $z$-axis. The configuration hence possesses zero net magnetic charge.

This MAP solution is similar to the MAP solutions of Kleihaus and Kunz [5]. Our MAP solution has $m = 1$, $s = 1$ and $z_0 = 2.4$ for $M = 40$, $N = 25$. When the accuracy in the $\theta$ grid is increased from $N = 15$ to $N = 25$, the two MAP solutions ($m = 1$, $s = 1$ and $m = 2$, $s = 0$) appear to approach each other from different direction. Questions of whether these two numerical solutions are the same configuration will be answered with more powerful computers and will be addressed elsewhere. It is also interesting to compare the energies of the two systems of MAPs and this will be calculated in another paper.

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# Some Comments on the Monopole, MAP, and MAC Solutions of the YMH Theory*

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**Abstract**

In this article, we would like to give some comments on the exact BPS one monopole solution, the MAP, and the MAC solutions. Firstly we notice that there are several different discrete sectors of the YMH vacuum as well as the Wu-Yang monopole system that can be parameterized by a single integer parameter $s$. The exact BPS one monopole solution connects the trivial YMH vacuum at small r to the Wu-Yang monopole solution at large r and it corresponds to the case when the parameter $s = 0$ at both small and large r. In the MAP and MAC solutions, the $\theta$-winding number is $m$ ($> 1$) and the $\phi$-winding number is $n$. Here we would like to comment that there always exist an equivalent form of solutions with $\theta$-winding number one and an integer parameter $s$ for the solutions with $\theta$-winding number $m$. By reducing the solutions to $m = 1$ and $s$, leads to a better understanding of the solutions. We also notice that asymptotic solutions with $m = 1$ and parameter $s$ will give better numerical accuracy.

# 1 Introduction

The SU(2) Yang-Mills-Higgs (YMH) field theory in $3 + 1$ dimensions, with the Higgs field in the adjoint representation are known to possess a large varieties of magnetic monopole configurations. The first finite energy monopole solution is the 't Hooft-Polyakov monopole solution [1] which is invariant under a U(1) subgroup of the local SU(2) gauge group. It has non zero Higgs mass and self-interaction and is a numerical solution. Later Prasad and Sommerfield [2] gave the closed form version of the 't Hooft-Polyakov monopole in the BPS limit. The YMH field

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theory with a unit magnetic charge and finite energy is spherically symmetric [1]-[2]. However multimonopole configurations with magnetic charges greater than unity cannot possess spherical symmetry [3] but at most axial symmetry [4].

So far exact monopole and multimonopoles solutions [2], [4] existed only in the Bogomol'nyi-Prasad-Sommerfield (BPS) limit. Outside this limit, when the Higgs field potential is non-vanishing, only numerical solutions are known. We have also shown that the ansatz of Ref.[5] possesses more exact multimonopole-antimonopole configurations in the BPS limit.

The axially symmetric monopole-antimonopole pair (MAP) of Kleihaus and Kunz [6], and the monopole-antimonopole chain (MAC) of Kleihaus et al. [7], [8] possess exact asymptotic solutions at both small and large distances. The connecting solutions at immediate distances are both numerical and non-BPS as they do not satisfy the Bogomol'nyi equation and are only solutions to the second order differential equations of motion. In this article, we would like to comment and give our view on the exact BPS one monopole solution of Prasad and Sommerfield [2], the MAP solution of Kleihaus and Kunz [6], and the MAC solutions of Kleihaus et al. [7], [8].

Firstly we notice that there are several different discrete sectors of the YMH vacuum as well as the Wu-Yang monopole system that can be parameterized by a single integer parameter $s$. The exact BPS one monopole solution of Ref. [2] connects the trivial YMH vacuum at small distances with the Wu-Yang monopole solution at large distances and it corresponds to the case when the parameter $s = 0$ at both small and large distances.

In the MAP and MAC solutions of Ref. [6]-[8], the $\theta$-winding number is $m$ ($> 1$) and the $\phi$-winding number is $n$. Here we would like to comment that there always exist an equivalent form of the solutions with winding number $m = 1$ and an integer parameter $s$ for all the solutions with $\theta$-winding number $m$. Hence by reducing the solutions to $m = 1$ and $s$, leads to a better understanding and accuracy of the connecting numerical solutions. The detail will be discussed in our work of Ref. [9] for the MAP case.

We also note that at small $r$ both the MAP and MAC solutions correspond to the trivial vacuum with $m = 1$ and $s = 0$. However at large $r$, the MAP solutions tend to a different sector of the vacuum with the parameter $s = 1, 2, 3, ...$ Hence the MAP solutions correspond to a one MAP when $s = 1$, a two MAP when $s = 2$, and so on with zero net topological magnetic charge. On the contrary, the MAC solutions tend to a different sector of the Wu-Yang monopole system at large $r$ when $s = 1, 2, 3,$ and hence possess topological magnetic charge of one when $n = 1$.

We briefly review the SU(2) Yang-Mills-Higgs field theory in the next section. In section 3, we give the MAP and MAC solutions with $m = 1$ and parameter $s$ and a discussion of these solutions. We also found that by calculating the numerical part of the solution with $m = 1$ gave higher accuracy of the monopole and antimonopole locations on the $z$-axis. In section 4, we present the BPS one monopole solution with $\theta$-winding number $m$ higher than one and we give the equivalent solutions when the $\theta$-winding $m$ is reduced to one. We end with some comments in section 5.

## 2 The SU(2) YMH Theory

The SU(2) YMH Lagrangian in 3+1 dimensions with non vanishing Higgs potential is given by

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}D^\mu\Phi^a D_\mu\Phi^a - \frac{1}{4}\lambda(\Phi^a\Phi^a - \frac{\mu^2}{\lambda})^2. \quad (1)$$

Here the Higgs field mass is $\mu$ and the strength of the Higgs potential is $\lambda$ which are constants. The vacuum expectation value of the Higgs field is $\mu/\sqrt{\lambda}$. The Lagrangian (1) is gauge invariant under the set of independent local SU(2) transformations at each space-time point. The covariant derivative of the Higgs field and the gauge field strength tensor are given respectively by

$$\begin{aligned} D_\mu\Phi^a &= \partial_\mu\Phi^a + g\epsilon^{abc}A^b_\mu\Phi^c, \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu. \end{aligned} \quad (2)$$

Since the gauge field coupling constant $g$ can be scaled away, we can set $g$ to one without any loss of generality. The metric used is $g_{\mu\nu} = (- + ++)$. The SU(2) internal group indices $a, b, c$ run from 1 to 3 and the spatial indices are $\mu, \nu, \alpha = 0, 1, 2$, and 3 in Minkowski space.

The equations of motion that follow from the Lagrangian (1) are

$$\begin{aligned} D^\mu F^a_{\mu\nu} &= \partial^\mu F^a_{\mu\nu} + \epsilon^{abc}A^{b\mu}F^c_{\mu\nu} = \epsilon^{abc}\Phi^b D_\nu\Phi^c, \\ D^\mu D_\mu\Phi^a &= -\lambda\Phi^a(\Phi^b\Phi^b - \frac{\mu^2}{\lambda}), \end{aligned} \quad (3)$$

and the Bogomol'nyi equation,

$$B^a_i \pm D_i\Phi^a = 0, \quad (4)$$

holds in the limit of vanishing $\mu$ and $\lambda$.

The tensor identified with the electromagnetic field, as proposed by 't Hooft [1] is

$$\begin{aligned} F_{\mu\nu} &= \hat{\Phi}^a F^a_{\mu\nu} - \epsilon^{abc}\hat{\Phi}^a D_\mu\hat{\Phi}^b D_\nu\hat{\Phi}^c, \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - \epsilon^{abc}\hat{\Phi}^a\partial_\mu\hat{\Phi}^b\partial_\nu\hat{\Phi}^c, \end{aligned} \quad (5)$$

where $A_\mu = \hat{\Phi}^a A^a_\mu$, the Higgs unit vector, $\hat{\Phi}^a = \Phi^a/|\Phi|$, and the Higgs field magnitude $|\Phi| = \sqrt{\Phi^a\Phi^a}$. The Abelian electric field is $E_i = F_{0i}$, and the Abelian magnetic field is $B_i = -\frac{1}{2}\epsilon_{ijk}F_{jk}$. We would also like to write the Abelian 't Hooft electromagnetic field as

$$F_{\mu\nu} = M_{\mu\nu} + H_{\mu\nu}, \quad (6)$$

$$\text{where} \quad M_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad H_{\mu\nu} = -\epsilon^{abc}\hat{\Phi}^a\partial_\mu\hat{\Phi}^b\partial_\nu\hat{\Phi}^c, \quad (7)$$

which we refer to as the gauge part and the Higgs part of the 't Hooft electomagnetic field respectively.

The topological magnetic current [10] is defined to be

$$k_\mu = \frac{1}{8\pi}\,\epsilon_{\mu\nu\rho\sigma}\,\epsilon_{abc}\,\partial^\nu\hat{\Phi}^a\,\partial^\rho\hat{\Phi}^b\,\partial^\sigma\hat{\Phi}^c, \quad (8)$$

which is also the topological current density of the system and the corresponding conserved topological magnetic charge is

$$\begin{aligned} M &= \int d^3x\ k_0 = \frac{1}{8\pi}\int \epsilon_{ijk}\epsilon^{abc}\partial_i\left(\hat{\Phi}^a\partial_j\hat{\Phi}^b\partial_k\hat{\Phi}^c\right) d^3x \\ &= \frac{1}{8\pi}\oint d^2\sigma_i\left(\epsilon_{ijk}\epsilon^{abc}\hat{\Phi}^a\partial_j\hat{\Phi}^b\partial_k\hat{\Phi}^c\right) \\ &= \frac{1}{4\pi}\oint d^2\sigma_i\ B_i. \end{aligned} \tag{9}$$

As mentioned by Arafune et al. [11], the magnetic charge $M$ is the total magnetic charge of the system if and only if the gauge field is non singular. If the gauge field is singular and carries Dirac string monopoles, then the total magnetic charge of the system is the total sum of the Dirac string monopoles and the monopoles carry by the Higgs field which is $M$.

# 3 The MAP and MAC Solutions

The magnetic ansatz of Ref.[5] can be generalized to include unit vectors of the internal space of higher than one $\theta$-winding number $m$ and $\phi$-winding number $n$,

$$\begin{aligned} A_i^a &= -\frac{1}{r}\psi_1(r,\theta)\hat{u}_\phi^a\hat{\theta}_i + \frac{1}{r}\psi_2(r,\theta)\hat{u}_\theta^a\hat{\phi}_i + \frac{1}{r}R_1(r,\theta)\hat{u}_\phi^a\hat{r}_i - \frac{1}{r}R_2(r,\theta)\hat{u}_r^a\hat{\phi}_i, \\ \Phi^a &= \frac{1}{r}\chi_1(r,\theta)\ \hat{u}_r^a + \frac{1}{r}\chi_2(r,\theta)\hat{u}_\theta^a. \end{aligned} \tag{10}$$

The spatial spherical coordinate orthonormal unit vectors, $\hat{r}_i$, $\hat{\theta}_i$, and $\hat{\phi}_i$ are defined by

$$\begin{aligned} \hat{r}_i &= \sin\theta\ \cos\phi\ \delta_{i1} + \sin\theta\ \sin\phi\ \delta_{i2} + \cos\theta\ \delta_{i3}, \\ \hat{\theta}_i &= \cos\theta\ \cos\phi\ \delta_{i1} + \cos\theta\ \sin\phi\ \delta_{i2} - \sin\theta\ \delta_{i3}, \\ \hat{\phi}_i &= -\sin\phi\ \delta_{i1} + \cos\phi\ \delta_{i2}. \end{aligned} \tag{11}$$

and the isospin coordinate orthonormal unit vectors, $\hat{u}_r^a$, $\hat{u}_\theta^a$, and $\hat{u}_\phi^a$ are defined by

$$\begin{aligned} \hat{u}_r^a &= \sin m\theta\ \cos n\phi\ \delta_1^a + \sin m\theta\ \sin n\phi\ \delta_2^a + \cos m\theta\ \delta_3^a, \\ \hat{u}_\theta^a &= \cos m\theta\ \cos n\phi\ \delta_1^a + \cos m\theta\ \sin n\phi\ \delta_2^a - \sin m\theta\ \delta_3^a, \\ \hat{u}_\phi^a &= -\sin n\phi\ \delta_1^a + \cos n\phi\ \delta_2^a. \end{aligned} \tag{12}$$

In this paper we take the MAP solutions to be a M-A-M-A-.....-M-A (MAPs) chain lying on the $z$-axis and the MAC solutions to be the M-A-M-A-........-M-A-M (MAPs-M) chain. Hence the MAP solutions possess zero net magnetic charge whereas the MAC solutions have a net magnetic charge of $n \leq 2$. The exact MAP asymptotic solutions of winding numbers $m$ and $n$ at small and large $r$ are [6]-[8]

$$\begin{aligned} \psi_1 &= \psi_2 = R_1 = R_2 = 0, \\ \chi_1 &= \xi r\cos m\theta, \qquad \chi_2 = -\xi r\sin m\theta, \quad \text{and} \end{aligned} \tag{13}$$

$$\begin{aligned} \psi_1(r,\theta) &= m, \quad \psi_2(r,\theta) = \frac{n\sin m\theta}{\sin\theta}, \\ R_1(r,\theta) &= 0, \quad R_2(r,\theta) = \frac{n(\cos m\theta - 1)}{\sin\theta}, \\ \chi_1(r,\theta) &= h + \xi r, \quad \chi_2(r,\theta) = 0, \end{aligned} \tag{14}$$

respectively, where $h$, $\xi$ are constants and $m$ an even integer. In the BPS limit $h = 0$, otherwise it is arbitrary. The exact MAC asymptotic solutions at small and large $r$ are [6]-[8]

$$\psi_1 = \psi_2 = R_1 = R_2 = \chi_1 = \chi_2 = 0, \quad \text{and} \tag{15}$$

$$\begin{aligned}
\psi_1(r,\theta) &= m, \quad \psi_2(r,\theta) = \frac{n \sin m\theta}{\sin\theta}, \\
R_1(r,\theta) &= 0, \quad R_2(r,\theta) = \frac{n(\cos m\theta - \cos\theta)}{\sin\theta}, \\
\chi_1(r,\theta) &= h + \xi r, \quad \chi_2(r,\theta) = 0,
\end{aligned} \tag{16}$$

respectively, where $m$ is an odd integer. In the BPS limit $h = n$ but it is however arbitrary outside this limit.

By using the relationship of the isospin unit vectors of winding number $m > 1$ with the isospin unit vectors of $\theta$-winding number one,

$$\begin{aligned}
\hat{u}_r^a &= \cos(m-1)\theta\ \hat{n}_r^a + \sin(m-1)\theta\ \hat{n}_\theta^a, \\
\hat{u}_\theta^a &= -\sin(m-1)\theta\ \hat{n}_r^a + \cos(m-1)\theta\ \hat{n}_\theta^a,
\end{aligned} \tag{17}$$

where $\hat{n}_r^a = \sin\theta\ \cos n\phi\ \delta_1^a + \sin\theta\ \sin n\phi\ \delta_2^a + \cos\theta\ \delta_3^a$, $\hat{n}_\theta^a = \cos\theta\ \cos n\phi\ \delta_1^a + \cos\theta\ \sin n\phi\ \delta_2^a - \sin\theta\ \delta_3^a$, we can reduced the MAP and MAC solutions of Eq. (13) - (16) to winding number $m = 1$. Hence the MAP asymptotic solutions with $m = 1$ is given by

$$\begin{aligned}
\psi_1 &= \psi_2 = R_1 = R_2 = 0, \\
\chi_1 &= \xi r\cos\theta, \qquad \chi_2 = -\xi r \sin\theta, \quad r \to 0 \ \text{ and}
\end{aligned} \tag{18}$$

$$\begin{aligned}
\psi_1 &= 2s, \quad \psi_2 = \frac{2n \sin s\theta \cos(s-1)\theta}{\sin\theta}, \\
R_1 &= 0, \quad R_2 = \frac{2n \sin s\theta \sin(s-1)\theta}{\sin\theta}, \\
\chi_1 &= (h+\xi r)\cos(2s-1)\theta, \quad \chi_2 = (h+\xi r)\sin(2s-1)\theta, \quad r \to \infty.
\end{aligned} \tag{19}$$

Similarly the MAC asymptotic solutions with $m = 1$ is given by

$$\begin{aligned}
\psi_1 &= \psi_2 = R_1 = R_2 = 0, \\
\chi_1 &= \xi r\cos\theta, \qquad \chi_2 = -\xi r \sin\theta, \quad r \to 0 \ \text{ and}
\end{aligned} \tag{20}$$

$$\begin{aligned}
\psi_1 &= 1 + 2s, \quad \psi_2 = \frac{\sin\theta + \sin 2s\theta \cos\theta}{\sin\theta}, \\
R_1 &= 0, \quad R_2 = \frac{\cos\theta - \cos 2s\theta\cos\theta}{\sin\theta}, \\
\chi_1 &= (h+\xi r)\cos 2s\theta, \quad \chi_2 = (h+\xi r)\sin 2s\theta, \quad r \to \infty.
\end{aligned} \tag{21}$$

at large $r$. The parameter $s$ is an integer. When $s = 0$ in Eq. (19), the solution is just the trivial YMH vacuum. The MAP asymptotic solution Eq. (19) is a pure gauge vacuum solution whereas the MAC asymptotic solution Eq. (21) is a Wu-Yang type monopole system of net magnetic charge $n$.

When $s = 1$ in Eq. (19), the MAP solution is a magnetic dipole, when $s = 2$, it is a two magnetic dipoles (M-A-M-A). Hence the general MAP solution is a $s$ magnetic dipoles solution. However when $s = 0$ in Eq. (21), it is the Wu-Yang solution and the MAC solution is just a finite energy one monopole solution [1], [2]. In the BPS limit, we get the exact one monopole solution of Ref. [2]. When $s = 1$, the MAC solution is the M-A-M chain, when $s = 2$, the monopole chain is M-A-M-A-M, and so on.

The purpose of reducing the MAP and MAC solutions to the $m = 1$ winding number is that when we compute for the numerical connecting solution between the two asymptotic solutions at small and large $r$, we notice that when we increased the accuracy of the numerical solutions, the positions of the monopole and antimonopole of the one dipole on the $z$-axis approaches a critical point on its left from the right when $m = 1$. In the case when $m = 2$, the positions of the magnetic poles approaches the critical point on its right from the left as the accuracy increases. If the exact connecting solution is unique, then the position of the magnetic poles lies on the critcal point inbetween these two sets of numerical solutions. Hence by running both $m = 1$ and $m = 2$ solutions will give a more accurate solution. The detail of this work is discussed in Ref. [9].

# 4 The exact BPS One Monopole

To solve for exact solutions, the ansatz (10) is substituted into the Bogomol'nyi equation (4). We find that the BPS exact one monopole solution with higher $\theta$-winding number and $\phi$-winding number is

$$\begin{aligned}
\psi_1(r,\theta) &= m \pm \frac{\xi r}{\sinh \xi r}, \quad \psi_2(r,\theta) = \frac{n \sin m\theta}{\sin\theta} \pm \frac{\xi r}{\sinh\xi r},\\
R_1(r,\theta) &= 0, \quad R_2(r,\theta) = \frac{n\cos m\theta - \cos\theta}{\sin\theta},\\
\chi_1(r,\theta) &= 1 - \frac{\xi r}{\tanh\xi r}, \quad \chi_2(r,\theta) = 0.
\end{aligned} \tag{22}$$

where $\xi$ is a constant. The solution (22) is non singular only when $m$ is odd and $n = 1$. Upon calculating for the non-Abelian magnetic field of solution (22), we get

$$B_i^a = \frac{1}{r^2}\left\{\left(1 - \frac{\xi^2 r^2}{\sinh^2\xi r}\right)\hat{u}_r^a \hat{r}_i - \frac{\xi r}{\sinh\xi r}\left(1 - \frac{\xi r}{\tanh\xi r}\right)(\hat{u}_\theta^a\hat{\theta}_i + \hat{u}_\phi^a\hat{\phi}_i)\right\}, \tag{23}$$

which is just the BPS one monopole magnetic field of higher winding number $m$. The energy or mass of the system is finite,

$$\mathcal{E} = \int dx^3 (\frac{1}{4}F_{ij}^a F_{ij}^a + \frac{1}{2}F_{0i}^a F_{0i}^a + \frac{1}{2}D_i\Phi^a D_i\Phi^a + \frac{1}{2}D_0\Phi^a D_0\Phi^a) = 4\pi\xi. \tag{24}$$

The Abelian gauge potential $A_\mu$ is non vanishing and is given by

$$A_\mu = \left\{\frac{\cos\theta - \cos m\theta}{r\sin\theta}\right\}\hat{\phi}_\mu. \tag{25}$$

Hence the gauge part of the Abelian magnetic field is

$$B_i \text{ (gauge part)} = \left(1 - \frac{m \sin m\theta}{\sin\theta}\right) \frac{\hat{r}_i}{r^2}, \tag{26}$$

which when $m = 3$ is the magnetic field of a zero length A-M-M-A chain at the origin along the $z$-axis. Hence the net magnetic charge is zero. The Higgs part of the Abelian magnetic field is

$$B_i \text{ (Higgs part)} = \frac{m \sin m\theta}{\sin\theta} \frac{\hat{r}_i}{r^2}, \tag{27}$$

and it corresponds to a M-A-M chain of zero length at the origin of net magnetic charge one.

By using Eq. (17), we can reduced the winding number $m$ of solution (22) to one. Hence solution (22) with $m = 1$ and $n = 1$ is given by

$$\begin{aligned}
\psi_1 &= 1 + 2s \pm \frac{\xi r}{\sinh \xi r}, \quad \psi_2 = \frac{\sin\theta + \sin 2s\theta \cos\theta}{\sin\theta} \pm \frac{\xi r \cos 2s\theta}{\sinh \xi r}, \\
R_1 &= 0, \quad R_2 = \frac{\cos\theta - \cos 2s\theta \cos\theta}{\sin\theta} \pm \frac{\xi r \sin 2s\theta}{\sinh \xi r}, \\
\chi_1 &= \left(1 - \frac{\xi r}{\tanh \xi r}\right) \cos 2s\theta, \quad \chi_2 = \left(1 - \frac{\xi r}{\tanh \xi r}\right) \sin 2s\theta. \qquad (28)
\end{aligned}$$

Solution (28) is equivalent to solution (22) of winding number $m = 2s + 1$. The exact BPS one monopole solution of Ref. [2] corresponds to solution (28) when $s = 0$. When $s = 1$, the solution is

$$\begin{aligned}
\psi_1 &= 3 \pm \frac{\xi r}{\sinh \xi r}, \quad \psi_2 = 2 + \cos 2\theta \left(1 \pm \frac{\xi r}{\sinh \xi r}\right), \\
R_1 &= 0, \quad R_2 = \sin 2\theta \left(1 \pm \frac{\xi r}{\sinh \xi r}\right), \\
\chi_1 &= \left(1 - \frac{\xi r}{\tanh \xi r}\right) \cos 2\theta, \quad \chi_2 = \left(1 - \frac{\xi r}{\tanh \xi r}\right) \sin 2\theta. \qquad (29)
\end{aligned}$$

This general exact one monopole BPS solution (29) possess finite energy. It is non singular at all space except at the origin when $r$ tends to zero. However at small $r$, the gauge field is a pure gauge with $B_i^a = 0$.

# 5 Comments

The MAP and MAC are finite energy monopoles chain solutions of net magnetic charge zero and $n \leq 2$ respectively. As the exact form of these solutions have not yet been found, highly accurate numerical solutions will be useful in studying their properties. By computing these numerical solutions in the $m = 1$ winding number forms do give more accurate results. We have the performed the numerical calculations of the MAP $s = 1$ one dipole [9]. The purchase of higher performing computers will enable us to perform more numerical calculations for higher value $s$ monopoles chain.

We have observed that for the MAP solutions when the small $r$ asymptotic solution is the trivial vacuum and the large $r$ asymptotic solution is the pure gauge solution of $s \geq 1$, then the MAP is a $s$ magnetic dipoles chain along the $z$-axis. As for the MAC solutions when the small $r$ asymptotic solution is the trivial vacuum and the large $r$ asymptotic solution is the Wu-Yang type monopole solution of $s \geq 1$, then the MAC is a $s$ magnetic dipoles plus one monopole chain along the $z$-axis.

The exact BPS one monopole solution has a pure gauge vacuum $s$ solution at small $r$ and a Wu-Yang type monopole $s$ solution at large $r$ and hence is just a one monopole solution.

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**nopole-antimonopole and vortex rings**

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e SU(2) Yang-Mills-Higgs theory supports the existence of monopoles, antimonopoles, and vortex rings. In this paper, we would like to sent new exact static antimonopole-monopole-antimonopole (A-M-A) configurations. The net magnetic charge of these configurations is ays −1, while the net magnetic charge at the origin is always +1 for all positive integer values of the solution's parameter *m*. However, when ncreases beyond 1, vortex rings appear coexisting with these AMA configurations. The number of vortex rings increases proportionally with value of *m*. They are located in space where the Higgs field vanishes along rings. We also show that a single-point singularity in the Higgs d does not necessarily correspond to a structureless 1-monopole at the origin but to a zero-size monopole-antimonopole-monopole (MAM) ıcture when the solution's parameter *m* is odd. This monopole is the Wu-Yang-type monopole and it possesses the Dirac string potential in Abelian gauge. These exact solutions are a different kind of Bogomol'nyi-Prasad-Sommerfield (BPS) solutions as they satisfy the first-order gomol'nyi equation but possess infinite energy due to a point singularity at the origin of the coordinate axes. They are all axially symmetrical ɔut the *z*-axis. ©2005 *American Institute of Physics*

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# MONOPOLE-ANTIMONOPOLE AND VORTEX RINGS

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**Abstract**

The SU(2) Yang-Mills-Higgs theory supports the existence of monopoles, antimonopoles, and vortex rings. In this paper, we would like to present new exact static antimonopole-monopole-antimonopole (A-M-A) configurations. The net magnetic charge of these configurations is always negative one, whilst the net magnetic charge at the origin is always positive one for all positive integer values of the solution's parameter $m$. However, when $m$ increases beyond one, vortex rings appear coexisting with these A-M-A configurations. The number of vortex rings increases proportionally with the value of $m$. They are located in space where the Higgs field vanishes along rings. We also show that a single point singularity in the Higgs field does not necessarily corresponds to a structureless 1-monopole at the origin but to a zero size monopole-antimonopole-monopole (MAM) structure when the solution's parameter $m$ is odd. This monopole is the Wu-Yang type monopole and it possesses the Dirac string potential in the Abelian gauge. These exact solutions are a different kind of BPS solutions as they satisfy the first order Bogomol'nyi equation but possess infinite energy due to a point singularity at the origin of the coordinate axes. They are all axially symmetrical about the z-axis.

## 1 Introduction

The SU(2) Yang-Mills-Higgs (YMH) field theory, with the Higgs field in the adjoint representation possesses magnetic monopole, multimonopole, antimonopoles, and vortex rings solutions [1]-[8]. The only spherically symmetric monopole solution is the unit magnetic charge monopole. The 't Hooft-Polyakov monopole solution with non zero Higgs mass and Higgs self-interaction is numerically, spherically symmetrical [1]-[2]. Multimonopole solutions possess at most axial symmetry [3].

The model with non vanishing Higgs vacuum expectation value but vanishing Higgs potential possesses exact monopole and multimonopole solutions which can be obtained by solving the first order Bogomol'nyi equations [11]. These solutions satisfying the Bogomol'nyi-Prasad-Sommerfield (BPS) limit possess minimal energies. Exact monopole and multimonopoles solutions exist in the BPS limit [2] - [3] whilst outside the BPS limit, when the Higgs field potential is non vanishing only numerical solutions are known. Asymmetric multimonopole solutions with no rotational symmetry are numerical solutions even in the BPS limit [4].

At present, the different exact configurations of monopoles found are the BPS multimonopole solutions of magnetic charges $M$ greater than unity with all the magnetic charges superimposed into a single point in space [3]. These superimposed multimonopole solutions possess axial and mirror symmetries. Following these works, finitely separated 1-monopoles were also constructed [4]-[5].

Numerical axially symmetric non-Bogomol'nyi monopole-antimonopole chain solutions were also found to exist both in the limit of a vanishing Higgs potential as well as in the presence of a finite Higgs potential. Recently, numerical BPS axially symmetric vortex rings solutions have also been reported [6].

We have reported on the existence of a different type of BPS static monopole-antimonopole solution in Ref.[7]. This solution which is exact and axially symmetric, represents two separate antimonopoles located at equal distances along the z-axis from a 1-monopole which is located at the origin. We have also shown that the extended ansatz of Ref.[7] possesses more multimonopole-antimonopole configurations, together with their anti-configurations [8]. These configurations possess either radial, axial, or mirror symmetries about the z-axis and they represent different combinations of monopoles, multimonopole, and antimonopoles.

In general, configurations of the YMH field theory with a unit magnetic charge are spherically symmetric [1], [2], whilst multimonopole configurations with magnetic charges greater than unity possess at most axial symmetry [3]. However we have emphasized in a recent work [9] that unit magnetic charge configurations are not necessarily spherically symmetric. By employing the ansatz of Ref.[7] we have found exact unit magnetic charge solution that does not even possess axial symmetry but only mirror symmetry about a vertical plane through the z-axis. However the converse is true and it has been shown that multimonopole solutions cannot possess spherical symmetry [10]. We would also like to mention that within the ansatz of Ref.[7], half-monopole solutions have also been reported [9].

In this paper we would like to present new static axially symmetric antimonopole-monopole-antimonopole (A-M-A) configurations of the SU(2) YMH theory with the Higgs field in the adjoint representation. Here the Higgs field vanishes either at points corresponding to antimonopoles or at rings corresponding to vortex loops. The net magnetic charge of these configurations is always negative one, whilst the net magnetic charge at the origin, $r = 0$, is always positive one for all positive integer values of the solution's parameter $m$. However, when $m$ increases beyond one, vortex rings appear coexisting with these A-M-A configurations. The number of vortex rings in the configuration is equal to $(m - 1)$ where $m \geq 1$. They are located horizontally in space where the Higgs field is zero along rings. Hence this family of solutions all lies in the topologically non trival sector with topogical charge negative one.

The two antimonopoles of the solutions are located at the two zeros of the Higgs field along the z-axis, whilst the Wu-Yang type 1-monopole is located at a point singularity of the Higgs field at the origin. We also show that this single point singularity in the Higgs field need not corresponds to a structureless 1-monopole at the origin but to a zero size monopole-antimonopole-monopole (MAM) structure when $m$ is odd. These exact solutions are a different kind of BPS solutions as they satisfy the first order Bogomol'nyi equation but possess infinite energy due to a point singularity at the origin of the coordinate axes.

We briefly review the SU(2) Yang-Mills-Higgs field theory and discussed the boundary conditions of these solutions in the section 2. We discussed the magnetic ansatz and its formulation in section 3 and present the solutions in section 4. We end with some comments in section 5.

## 2 The SU(2) Yang-Mills-Higgs Theory

The SU(2) group admits the triplet Yang-Mills gauge fields potential $A_\mu^a$ which when coupled to a scalar Higgs triplets field $\Phi^a$ in 3+1 dimensions gives the SU(2) YMH theory [12]. The index $a$ is the SU(2) internal space index and for a given $a$, $\Phi^a$ is a scalar whereas $A_\mu^a$ is a vector under Lorentz transformation. The SU(2) YMH Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2}D^\mu\Phi^a D_\mu\Phi^a - \frac{1}{4}\lambda(\Phi^a\Phi^a - \frac{\mu^2}{\lambda})^2, \tag{1}$$

where the Higgs field mass, $\mu$, and the strength of the Higgs potential, $\lambda$, are constants. The vacuum expectation value of the Higgs field is then $\mu/\sqrt{\lambda}$. The Lagrangian (1) is gauge invariant under the set of independent local SU(2) transformations at each space-time point. The covariant derivative of the Higgs field and the gauge field strength tensor are given respectively by

$$\begin{aligned} D_\mu\Phi^a &= \partial_\mu\Phi^a + \epsilon^{abc}A_\mu^b\Phi^c, & (2)\\ F_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc}A_\mu^b A_\nu^c. & (3) \end{aligned}$$

Since the gauge field coupling constant $g$ can be scaled away, we can set $g$ to one without any loss of generality. The metric used is $g_{\mu\nu} = (- + ++)$. The SU(2) internal group indices $a, b, c$ run from 1 to 3 and the spatial indices are $\mu, \nu, \alpha = 0, 1, 2,$ and 3 in Minkowski space.

The equations of motion that follow from the Lagrangian (1) are

$$
\begin{aligned}
D^{\mu}F^a_{\mu\nu} &= \partial^{\mu}F^a_{\mu\nu} + \epsilon^{abc}A^{b\mu}F^c_{\mu\nu} = \epsilon^{abc}\Phi^b D_\nu \Phi^c, \\
D^{\mu}D_{\mu}\Phi^a &= -\lambda\Phi^a(\Phi^b\Phi^b - \frac{\mu^2}{\lambda}).
\end{aligned} \tag{4}
$$

The tensor to be identified with the Abelian electromagnetic field, as proposed by 't Hooft [1], [13] is

$$
\begin{aligned}
F_{\mu\nu} &= \hat{\Phi}^a F^a_{\mu\nu} - \epsilon^{abc}\hat{\Phi}^a D_\mu\hat{\Phi}^b D_\nu\hat{\Phi}^c \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu - \epsilon^{abc}\hat{\Phi}^a\partial_\mu\hat{\Phi}^b\partial_\nu\hat{\Phi}^c,
\end{aligned} \tag{5}
$$

where $A_\mu = \hat{\Phi}^a A^a_\mu$, $\hat{\Phi}^a = \Phi^a/|\Phi|$, $|\Phi| = \sqrt{\Phi^a\Phi^a}$. Hence the Abelian electric field is $E_i = F_{0i}$, and the Abelian magnetic field is $B_i = -\frac{1}{2}\epsilon_{ijk}F_{jk}$, where the indices, $i, j, k = 1, 2, 3$. The topological magnetic current, which is also the topological current density of the system is [13]

$$
k_\mu = \frac{1}{8\pi}\ \epsilon_{\mu\nu\rho\sigma}\ \epsilon_{abc}\ \partial^\nu\hat{\Phi}^a\ \partial^\rho\hat{\Phi}^b\ \partial^\sigma\hat{\Phi}^c. \tag{6}
$$

Therefore the corresponding conserved topological magnetic charge is

$$
\begin{aligned}
M &= \int d^3x\ k_0 = \frac{1}{8\pi}\int \epsilon_{ijk}\epsilon^{abc}\partial_i\left(\hat{\Phi}^a\partial_j\hat{\Phi}^b\partial_k\hat{\Phi}^c\right) d^3x \\
&= \frac{1}{8\pi}\oint d^2\sigma_i\left(\epsilon_{ijk}\epsilon^{abc}\hat{\Phi}^a\partial_j\hat{\Phi}^b\partial_k\hat{\Phi}^c\right) \\
&= \frac{1}{4\pi}\oint d^2\sigma_i\ B_i.
\end{aligned} \tag{7}
$$

Our work is restricted to the static case where $A^a_0 = 0$ with massless Higgs field and vanishing self-interaction. The magnitude of the Higgs field vanishes as $1/r$ at large $r$. However, this does not affect the Abelian magnetic field of the solutions as this magnetic field depends only on the unit vector of the Higgs field. It is in this limit that the solutions are solved using both the second order Euler-Lagrange equations and the Bogomol'nyi equations, $B^a_i \pm D_i\Phi^a = 0$. The $\pm$ sign corresponds to monopoles and antimonopoles respectively for the usual BPS solutions [13]. In our case, the A-M-A configuration is solved with the + sign and its anti-configuration, that is, the M-A-M configuration is solved with the − sign [8].

# 3 The Ansatz and its Formulation

The static gauge fields and Higgs field which will lead to the axially symmetric vortex rings solutions are given respectively by [7]

$$\begin{aligned} A_\mu^a &= \frac{1}{r}\psi(r)\left(\hat{\theta}^a\hat{\phi}_\mu - \hat{\phi}^a\hat{\theta}_\mu\right) + \frac{1}{r}R(\theta)\left(\hat{\phi}^a\hat{r}_\mu - \hat{r}^a\hat{\phi}_\mu\right), \\ \Phi^a &= \Phi_1\,\hat{r}^a + \Phi_2\,\hat{\theta}^a, \end{aligned} \tag{8}$$

where $\Phi_1 = \frac{1}{r}\psi(r)$, $\Phi_2 = \frac{1}{r}R(\theta)$. The spherical coordinate orthonormal unit vectors, $\hat{r}^a$, $\hat{\theta}^a$, and $\hat{\phi}^a$ are defined by

$$\begin{aligned} \hat{r}^a &= \sin\theta\,\cos\phi\;\delta_1^a + \sin\theta\,\sin\phi\;\delta_2^a + \cos\theta\delta_3^a, \\ \hat{\theta}^a &= \cos\theta\,\cos\phi\;\delta_1^a + \cos\theta\,\sin\phi\;\delta_2^a - \sin\theta\;\delta_3^a, \\ \hat{\phi}^a &= -\sin\phi\;\delta_1^a + \cos\phi\;\delta_2^a, \end{aligned} \tag{9}$$

where $r = \sqrt{x^i x_i}$, $\theta = \cos^{-1}(x_3/r)$, and $\phi = \tan^{-1}(x_2/x_1)$. The gauge field strength tensor and the covariant derivative of the Higgs field are given respectively by

$$\begin{aligned} F_{\mu\nu}^a &= \frac{1}{r^2}\hat{r}^a\left\{\dot{R} + R\cot\theta + 2\psi - \psi^2\right\}(\hat{\phi}_\mu\hat{\theta}_\nu - \hat{\phi}_\nu\hat{\theta}_\mu) \\ &+ \frac{1}{r^2}\hat{\theta}^a\left\{R(1-\psi)\right\}(\hat{\phi}_\mu\hat{\theta}_\nu - \hat{\phi}_\nu\hat{\theta}_\mu) \\ &+ \frac{1}{r^2}\left\{\hat{r}^a R(1-\psi) + \hat{\theta}^a(r\psi' + R\cot\theta - R^2)\right\}(\hat{r}_\mu\hat{\phi}_\nu - \hat{r}_\nu\hat{\phi}_\mu) \\ &+ \frac{1}{r^2}\hat{\phi}^a\left\{-(r\psi' + \dot{R})\right\}(\hat{r}_\mu\hat{\theta}_\nu - \hat{r}_\nu\hat{\theta}_\mu), \end{aligned} \tag{10}$$

$$\begin{aligned} D_\mu\Phi^a &= \frac{1}{r^2}\left\{\hat{r}^a(r\psi' - \psi - R^2) - \hat{\theta}^a R(1-\psi)\right\}\hat{r}_\mu \\ &+ \frac{1}{r^2}\left\{-\hat{r}^a R(1-\psi) + \hat{\theta}^a(\dot{R} + \psi - \psi^2)\right\}\hat{\theta}_\mu \\ &+ \frac{1}{r^2}\left\{\hat{\phi}^a(\psi - \psi^2 + R\cot\theta - R^2)\right\}\hat{\phi}_\mu. \end{aligned} \tag{11}$$

Here prime means $\partial/\partial r$ and dot means $\partial/\partial\theta$. The gauge fixing condition that we used here is the radiation or Coulomb gauge, $\partial^i A_i^a = 0$, $A_0^a = 0$.

The ansatz (8) is substituted into the equations of motion (4) as well as the Bogomol'nyi equations with the positive sign and the resulting equations of motion are just two first order differential equations,

$$r\psi' + \psi - \psi^2 = -p, \tag{12}$$

$$\dot{R} + R\cot\theta - R^2 = p, \tag{13}$$

where $p$ is an arbitary constant. Eq.(12) is exactly solvable for all real values of $p$ and the integration constant can be scaled away by letting $r \rightarrow r/c$, where $c$ is the arbitrary integration constant. Hence without any loss in generality, $c$ is set to unity. In order to obtain solutions of $\psi$ with $(2m+1)$ powers of $r$ we can write $p = m(m+1)$ where $m$ is real. By doing so, the solutions of the Riccati equation (13) can be exactly solved in terms of the Legendre functions of the first and second kind. For the solutions of Eq.(13) to be regular along the z-axis, we require $R(\theta)$ to vanish when $\theta = 0$ and $\theta = \pi$. To achieve these boundary conditions, the integration constant of Eq.(13) is set to zero and $m$ is restricted to take integer values. The solutions for $\psi$ and $R$ are then given respectively by

$$\begin{aligned} \psi(r) &= \frac{(m+1) - mr^{2m+1}}{1 + r^{2m+1}}, \\ R(\theta) &= (m+1)\left\{\cot\theta - \frac{P_{m+1}(\cos\theta)}{P_m(\cos\theta)}\csc\theta\right\}, \end{aligned} \tag{14}$$

where $P_m$ is the Legendre polynomial of degree $m$, and $m = 0, 1, 2, 3, \ldots$. Hence the boundary conditions of the solutions, Eq.(14), are $\psi(0) = m+1$, $\psi(\infty) = -m$, $R(0) = R(\pi) = 0$.

In the BPS limit, the energy can be written in the form

$$E = \mp \int \partial_i (B_i^a \Phi^a)\, d^3x + \int \frac{1}{2}(B_i^a \pm D_i\Phi^a)^2\, d^3x$$

$$= \mp \int \partial_i (B_i^a \Phi^a)\, d^3x = 4\pi M \frac{\mu}{\sqrt{\lambda}}, \tag{15}$$

where $M$ is the "topological charge" when the vacuum expectation value of the Higgs field, $\frac{\mu}{\sqrt{\lambda}}$ is non zero coupled with some non-trivial topological structure of the fields at large $r$.

The energy density $\partial_i(B_i^a\Phi^a)$ of the non-Abelian system is finite everywhere and vanishes as $1/r^2$ at large $r$ except at the origin $r = 0$ due to the presence of a point singularity there and along singular planes where $P_m(\cos\theta)$ vanishes. The two antimonopoles which are regular antimonopoles and the vortex rings are located at points and rings where the Higgs field, $r\Phi^a$, vanishes respectively. The monopole at the origin is of a different nature and is located where the Higgs field is singular. In the Abelian gauge, this monopole carries a Dirac string singularity.

The topological charge is also related to another gauge invariant quantity of the system as given by Eq.(7),

$$M_\infty = \frac{1}{8\pi} \oint d^2\sigma_i \left( \epsilon_{ijk}\epsilon^{abc}\hat{\Phi}^a \partial_j \hat{\Phi}^b \partial_k \hat{\Phi}^c \right)\Big|_{r\to\infty}. \tag{16}$$

The magnetic charges enclosed by the sphere at infinity can be associated with the zeros of the Higgs field, $r\Phi^a$ and at points where $\hat{\Phi}^a$ becomes indeterminate. The positions of the two antimonopole do correspond to the two point zeros of the Higgs field in the A-M-A solutions. However the Wu-Yang type 1-monopole is not located at the zeros of the Higgs field but at the origin of the coordinate axes where the Higgs field is singular.

From the ansatz (8), $A_\mu = \hat{\Phi}^a A_\mu^a = 0$. Hence from Eq.(5), the Abelian electric field is zero and the Abelian magnetic field is independent of the gauge fields $A_\mu^a$. To calculate for the 't Hooft Abelian magnetic field $B_i$, we rewrite the Higgs field of Eq.(8) from the spherical to the Cartesian coordinate system, [6]-[8]

$$\begin{aligned} \Phi^a &= \Phi_1\, \hat{r}^a + \Phi_2\, \hat{\theta}^a + \Phi_3\, \hat{\phi}^a \\ &= \tilde{\Phi}_1\, \delta^{a1} + \tilde{\Phi}_2\, \delta^{a2} + \tilde{\Phi}_3\, \delta^{a3} \end{aligned} \tag{17}$$

$$\begin{aligned}
\text{where} \quad \tilde{\Phi}_1 &= \sin\theta\cos\phi\ \Phi_1 + \cos\theta\cos\phi\ \Phi_2 - \sin\phi\ \Phi_3 = |\Phi|\cos\alpha\sin\beta \\
\tilde{\Phi}_2 &= \sin\theta\sin\phi\ \Phi_1 + \cos\theta\sin\phi\ \Phi_2 + \cos\phi\ \Phi_3 = |\Phi|\cos\alpha\cos\beta \\
\tilde{\Phi}_3 &= \cos\theta\ \Phi_1 - \sin\theta\ \Phi_2 = |\Phi|\sin\alpha. \qquad (18)
\end{aligned}$$

The Higgs unit vector is then simplified to

$$\begin{aligned}
\hat{\Phi}^a &= \cos\alpha\sin\beta\ \delta^{a1} + \cos\alpha\cos\beta\ \delta^{a2} + \sin\alpha\ \delta^{a3}, \qquad (19) \\
\text{where,} \quad \sin\alpha &= \frac{\psi\cos\theta - R\sin\theta}{\sqrt{\psi^2 + R^2}}, \quad \beta = \frac{\pi}{2} - \phi,
\end{aligned}$$

and the Abelian magnetic field is found to reduce to only the $\hat{r}_i$ and $\hat{\theta}_i$ components,

$$\begin{aligned}
B_i &= \frac{1}{r^2\sin\theta}\left\{\frac{\partial\sin\alpha}{\partial\theta}\frac{\partial\beta}{\partial\phi} - \frac{\partial\sin\alpha}{\partial\phi}\frac{\partial\beta}{\partial\theta}\right\}\hat{r}_i \\
&+ \frac{1}{r\sin\theta}\left\{\frac{\partial\sin\alpha}{\partial\phi}\frac{\partial\beta}{\partial r} - \frac{\partial\sin\alpha}{\partial r}\frac{\partial\beta}{\partial\phi}\right\}\hat{\theta}_i \\
&+ \frac{1}{r}\left\{\frac{\partial\sin\alpha}{\partial r}\frac{\partial\beta}{\partial\theta} - \frac{\partial\sin\alpha}{\partial\theta}\frac{\partial\beta}{\partial r}\right\}\hat{\phi}_i, \\
&= -\frac{1}{r^2\sin\theta}\left\{\frac{\partial\sin\alpha}{\partial\theta}\right\}\hat{r}_i + \frac{1}{r\sin\theta}\left\{\frac{\partial\sin\alpha}{\partial r}\right\}\hat{\theta}_i. \qquad (20)
\end{aligned}$$

Since $\sin\alpha$ is a non singular function except at the points where the Higgs field, $r\Phi^a$ vanishes, the 't Hooft magnetic field is regular everywhere except at the locations of the A-M-A and the vortex rings.

By noticing that the magnetic field Eq.(20) can also be written as

$$\begin{aligned}
B_i &= \epsilon_{ijk}\ \partial^j(\sin\alpha)\ \partial^k\beta, \\
&= \epsilon_{ijk}\ \partial^j(\sin\alpha\ \partial^k\beta), \qquad (21)
\end{aligned}$$

we found that a suitable Maxwell four-vector gauge potential for this 't Hooft magnetic field is

$$\begin{aligned}
\mathcal{A}_0 &= 0, \\
\mathcal{A}_i &= (\sin\alpha - 1)\partial_i\beta = -\frac{(\sin\alpha - 1)}{r\sin\theta}\hat{\phi}_i. \qquad (22)
\end{aligned}$$

This gauge potential also satisfies the Coulomb gauge condition, $\partial^i \mathcal{A}_i = 0$. The function $\sin\alpha$ is a non singular function of $r$ and $\theta$, and it is not smooth and discontinuous only when the profile function $R(\theta)$ is singular. When $m = 0$, the gauge potential Eq.(22) is just the usual Dirac string potential and it is singular along the negative z-axis. However when $m > 0$, the gauge potential $\mathcal{A}_i$ possesses a broken Dirac string singularity. This Dirac string singularity extends from the origin to $z = -\sqrt[2m+1]{\frac{m+1}{m}}$ along the negative z-axis and from $z = \sqrt[2m+1]{\frac{m+1}{m}}$ to infinity along the positive z-axis. The gauge potential, $\mathcal{A}_i$, is only discontinuous at values of $\theta$ when $R(\theta)$ is singular.

From Eq.(21), it is obvious that the magnetic field is always perpendicular to the gradient of $\sin\alpha$. Hence the magnetic field lines lie on the line $\sin\alpha = k$, $-1 < k < 1$, and $\phi =$ constant. By plotting $\sin\alpha = k$ on a vertical plane through the origin; we manage to draw the magnetic field lines for the configurations when $m = 1$, Fig.(2); $m = 2$, Fig.(6); and $m = 3$, Fig.(8).

Defining the Abelian field magnetic flux as

$$\Omega = 4\pi M = \oint d^2\sigma_i B_i = \int B_i(r^2 \sin\theta d\theta)\hat{r}_i \; d\phi, \tag{23}$$

the magnetic charge enclosed by the sphere at infinity, $M_\infty$, is calculated to be,

$$\begin{aligned} M_\infty &= -\frac{1}{2}\sin\alpha\bigg|_{0,r\to\infty}^{\pi} = -1, \quad \text{when} \;\; m = 1, 2, 3, \ldots \\ &= 1, \quad \text{when} \;\; m = 0. \end{aligned} \tag{24}$$

From Eq. (24), we can conclude that the total magnetic charge $M$ of these axially symmetric solutions does not depend on the degree of the Legendre polynomial when $m > 0$. Hence the net magnetic charge of the system when $m > 0$ is always negative one. By letting $M_0$ to be the net magnetic charge when the radius of the enclosing sphere tends to zero at the origin, we get,

$$M_0 = -\frac{1}{2}\sin\alpha\bigg|_{0,r\to 0}^{\pi} = 1, \quad m = 0, 1, 2, 3, \ldots. \tag{25}$$

Similiarly, the net magnetic charge, $M_0$, at the point singularity of the solution is independent of the value of $m$. In fact, it is true that for positive non zero $m$, when $r < \sqrt[2m+1]{\frac{m+1}{m}}$, the topological magnetic charge is one, and when $r > \sqrt[2m+1]{\frac{m+1}{m}}$, the topological magnetic charge is negative one. Hence there is a 1-monopole located at $r = 0$ and two antimonopoles located along the z-axis at $r = \sqrt[2m+1]{\frac{m+1}{m}}$.

We also notice that we can write the net magnetic flux per $4\pi$ sterad passing through the spherical surface of a partial enclosing sphere of radius $r$, sustaining an angle $\theta$ at the origin with the positive z-axis as

$$\begin{aligned} M_r(\theta) &= -\left.\frac{(\psi(r)\cos\theta - R(\theta)\sin\theta)}{2\sqrt{\psi^2(r) + R^2(\theta)}}\right|_0^\theta \\ &= \frac{1}{2}\left\{\frac{\psi}{|\psi|} - \sin\alpha\right\}. \end{aligned} \tag{26}$$

# 4 Monopole, Antimonopoles, and Vortex Rings

The first member of this series of axially symmetric solutions is when $m = 0$. As discussed in our previous work [7], this solution is the Wu-Yang type monopole located at $r = 0$. This radially symmetric monopole with its magnetic field, $B_i = \frac{1}{r^2}\hat{r}_i$, has the vector potential $\mathcal{A}_i$ given by Eq.(22). This is just the Dirac string gauge potential, $\mathcal{A}_\mu = \frac{1}{r}\tan(\frac{1}{2}\theta)\hat{\phi}_\mu$, which is singular along the negative z-axis.

When $m = 1$, the configuration is the second member of the axially symmetric monopole solutions. This configuration is similar to the A1 solution of Ref.[7] and [8] with gauge potentials and Higgs field given by

$$\begin{aligned} A^a_\mu &= \frac{1}{r}\left\{\frac{2 - r^3}{1 + r^3}\right\}\left(\hat{\theta}^a\hat{\phi}_\mu - \hat{\phi}^a\hat{\theta}_\mu\right) + \frac{1}{r}\tan\theta\left(\hat{\phi}^a\hat{r}_\mu - \hat{r}^a\hat{\phi}_\mu\right), \\ \Phi^a &= \frac{1}{r}\left\{\frac{2 - r^3}{1 + r^3}\right\}\hat{r}^a + \frac{1}{r}\tan\theta\ \hat{\theta}^a. \end{aligned} \tag{27}$$

It was first thought to consist of a 1-monopole at $r = 0$, surrounded by two antimonopoles located at the point zeros of the Higgs field at $z = \pm\sqrt[3]{2} = \pm 1.2599$.

However, a closer study of this solution reveals that the 1-monopole actually has a zero size MAM structure along the z-axis and hence possess a net unit magnetic charge. This MAM structure can be read from the plots of Eq.(26), $M_r(\theta)$ versus $\theta$ for the cases of $r \to 0$ and $r \to \infty$, see Fig.(1). The plot at $r \to \infty$ indicates that there is zero flux through the spherical shell at infinity when $\theta \neq \frac{\pi}{2}$ rad. Hence all the flux at infinity is radially inwards along the equatorial plane towards the origin, $r = 0$, giving a net topological charge of negative one for the $m = 1$ configuration. The radial component of the magnetic field at large $r$ is just a Dirac delta function of $\theta$, given by $B_r = -\frac{2}{r^2 \sin\theta}\delta(\theta - \frac{\pi}{2})$. Hence the singularity of the solution $R(\theta)$ at the equatorial plane $\theta = \pi/2$ gives rise to a Dirac delta function singularity in the magnetic field. The antimonopole at the positive z-axis interacts with the nearest MAM 1-monopole at $r = 0$ to form a dipole pair and similarly the antimonopole at the negative z-axis interacts with the other MAM 1-monopole at $r = 0$ to form the other dipole pair leaving behind an antimonopole at the origin, see Fig.(2). At large distances all the magnetic field above and below the equatorial plane are being screened off by the two dipole pairs along the z-axis leaving behind a Dirac delta function magnetic field along the plane of singularity, $\theta = \pi/2$.

At finite $r < \sqrt[3]{2}$, the radial component of the magnetic field is a regular function of $r$ and $\theta$ but not smooth at $\theta = \pi/2$. In fact, at $\theta = \pi/2$, $B_r$ possesses a negative Dirac delta function singularity indicating an antimonopole at the center of the composite 1-monopole. Fig.(1) shows that at small $r$, the net flux through the upper $(0 < \theta < \frac{\pi}{2})$ and lower $(\frac{\pi}{2} < \theta < \pi)$ spherical shell is $+4\pi$ each and the flux through the circle at constant $r$ and $\theta = \frac{\pi}{2}$ is $-4\pi$, hence indicating a MAM structure for the 1-monopole at the origin.

By plotting the magnetic field lines of this configuration we can confirm that at large $r$, all the magnetic field lies in the equatorial plane and is pointing radially inwards as the net magnetic charge $M_\infty$ is $-1$. A plot of the magnetic field lines is shown in Fig.(2). Hence the pole at the center of the composite monopole is an antimonopole surrounded by two 1-monopoles at zero range from each other and yet they do not annihilate each. The antimonopoles situated at $z = \pm\sqrt[3]{2}$ form dipole pairs with the nearest 1-monopoles of the MAM structure, thus screening off all the magnetic field above and below the equatorial plane at $r$ infinity. There is no vortex ring in this configuration. The Abelian gauge potential, $\mathcal{A}_\mu = -\frac{(\sin\alpha - 1)}{r\sin\theta}\hat{\phi}_\mu$, possesses a Dirac string singularity along negative $z$-axis for $0 < r < \sqrt[3]{2}$ and along the positive $z$-axis for $r > \sqrt[3]{2}$ to infinity.

The vortex ring appears when $m = 2$, that is, when the gauge field potentials and Higgs field are respectively

$$\begin{aligned} A^a_\mu &= \frac{1}{r}\left\{\frac{3-2r^5}{1+r^5}\right\}\left(\hat{\theta}^a\hat{\phi}_\mu - \hat{\phi}^a\hat{\theta}_\mu\right) + \frac{1}{r}\left\{\frac{6\cos\theta\sin\theta}{3\cos^2\theta - 1}\right\}\left(\hat{\phi}^a\hat{r}_\mu - \hat{r}^a\hat{\phi}_\mu\right), \\ \Phi^a &= \frac{1}{r}\left\{\frac{3-2r^5}{1+r^5}\right\}\hat{r}^a + \frac{1}{r}\left\{\frac{6\cos\theta\sin\theta}{3\cos^2\theta - 1}\right\}\hat{\theta}^a. \end{aligned} \tag{28}$$

The plots of the magnetic flux, Eq.(26), versus $\theta$, for values of $r \longrightarrow 0$ and $r$ at infinity, Fig.(3), reveal that the 1-monopole at $r = 0$ has a MAMAM structure. However only the monopole at the center has unit charge. All the other four poles have charge less than unity. Hence there exist a vortex point both above and below the 1-monopole.

The two outer regular antimonopoles are located at the two point zeros of the Higgs field at $z = \pm\sqrt[5]{3/2} = \pm 1.0845$, Fig.(4), and the vortex ring is located along the ring of radius 1.0845 on the equatorial plane where the Higgs field vanishes, Fig.(5). The magnetic field lines of this one vortex ring solution is shown in Fig.(6).

The two vortex rings solution is the next solution of this series of axially symmetric monopole configurations with parameter $m = 3$. The gauge field potentials and Higgs field are respectively given by

$$A^a_\mu = \frac{1}{r}\left\{\frac{4-3r^7}{1+r^7}\right\}\left(\hat{\theta}^a\hat{\phi}_\mu - \hat{\phi}^a\hat{\theta}_\mu\right) + \frac{3\tan\theta}{r}\left\{\frac{5\cos^2\theta - 1}{5\cos^2\theta - 3}\right\}\left(\hat{\phi}^a\hat{r}_\mu - \hat{r}^a\hat{\phi}_\mu\right),$$

$$\Phi^a = \frac{1}{r}\left\{\frac{4-3r^7}{1+r^7}\right\}\hat{r}^a + \frac{3\tan\theta}{r}\left\{\frac{5\cos^2\theta-1}{5\cos^2\theta-3}\right\}\hat{\theta}^a. \qquad (29)$$

As usual the two point antimonopoles are situated at the two point zeros, $z = \pm\sqrt[7]{4/3} = \pm 1.0420$, of the Higgs field. The two vortex rings are located horizontally at $r = 1.0420$ and $\theta = 1.1071, (\pi - 1.1071)$ rad. Again from the plots of magnetic flux, $M_r(\theta)|_{r\to 0}$ and $M_r(\theta)|_{r\to\infty}$, Fig.(7), of Eq.(26), together with the plot of the magnetic field lines, Fig.(8), we can conclude that the structure of the composite 1-monopole at the origin is MAMAMAM, with an antimonopole at the center. The three poles in the center, MAM, possess unity charge whereas the other four poles possess charge less than unity. Hence there exist a vortex point both above and below the MAM 1-monopole. Hence by induction, we conclude that the number of A and M "poles" in the composite monopole is equal to $2m+1$, and when $m$ is even, the pole in the center of the structure is a monopole and when $m$ is odd, we have an antimonopole or a MAM 1-monopole in the center of the structure [6]. The number of vortex rings in the solution increases with $m$ and is equal to $(m-1)$.

# 5 Comments

We have obtained exact axially symmetric A-M-A configurations of the SU(2) YMH theory which are characterized by a positive integer parameter $m$. The 1-monopole which is located at the origin, $r = 0$ where the Higgs field is singular, is a Wu-Yang type monopole. The two regular outer antimonopoles are located at the two point zeros of the Higgs field along the z-axis at $z = \pm\sqrt[2m+1]{(m+1)/m}$. When the parameter $m$ exceeds unity, vortex rings start to appear around the z-axis. The number of vortex rings in the solution is equal to $(m-1)$.

Further investigations reveal that the 1-monopole at the origin possesses structure. It corresponds to a zero size composite monopole with its axis lying along the z-axis. By induction we conclude that the number of "poles" in the composite monopole is given by $2m+1$. When $m$ is even, the center of the structure corresponds to a 1-monopole M and when $m$ is odd, it corresponds to an antimonopole or a MAM 1-monopole. We have analysed the solutions for the cases of $m=0,1,2$, and 3, with 1-monopole given by M, MAM, MAMAM, and MAMAMAM, respectively. The MA and AM above and below the 1-monopole when $m=2$ and 3, can be thought of as vortex point as the magnetic charges of these "poles" are less than unity.

There are two types of singularities in solutions (14). The point singularity at the origin, $r=0$, gives rise to a Wu-Yang type monopole, M or MAM. This monopole posssesses the usual Dirac string potential in the Abelian gauge when $m=0$. However when $m=1,2,3,\ldots$, the Dirac string is broken into two parts. The string stretches from $z=0$ to $z=-\sqrt[2m+1]{\frac{m+1}{m}}$ along the negative z-axis and from $z=\sqrt[2m+1]{\frac{m+1}{m}}$ to positive infinity along the positive z-axis.

The singularities in $R(\theta)$ when $P_m(\cos\theta)=0$ give rise to plane singularities. The number of singular planes in the solution is equal to $m$. Hence when $m=1$, the singular plane is the equatorial plane. The Abelian magnetic field possesses a negative Dirac delta function singularity along this plane, $B_r=-\frac{2}{r^2\sin\theta}\delta(\theta-\frac{\pi}{2})$. Similarly when $m=2$, the singular planes are $\theta=0.9553$ and 2.1863 rad and when $m=3$, the singular planes are $\theta=0.6847$, 0, and 2.4569 rad. In all these solutions, the Abelian magnetic fields possess negative Dirac delta function singularity along these planes as the Abelian gauge potentials are discontinuous at these values of $\theta$.

Numerical static axially symmetric M-A-M-... chain at finite poles separations has also been discussed in Ref.[6]. These numerical solutions belong to the topologically trivial sector when the total number of poles and antipoles is even and to the topological unit sector when the total number of poles and antipoles is odd.

We have only managed to find odd total number of poles and antipoles in our solutions. Similar to the results of Ref.[6], we have a monopole at the center of the composite 1-monopole when $m$ is even and an antimonopole in the center when $m$ is odd. Also similar is that our solutions have zero magnetic dipole moment as the number of poles in our solutions is odd.

Unlike the monopole solutions of Ref.[6], our A-M-A poles here are of unit charge only. We did not manage to get monopoles and antimonopoles of charge equal to two units for our axially symmetric monopoles solutions. In fact we have not found any M-monopoles with finite separations when $|M| \geq 2$.

We would also like to mention that for every monopoles, antimonopole, vortex rings solutions that we have discussed so far, there always exist an anti-configuration of the configurations discussed. This can be done by changing the $\phi$ winding number in the ansatz (8) from one to $-1$ and solving the Bogomol'nyi equation with the negative sign [8].

We would also like to mention that one-half topological magnetic charge monopole is obtained when the parameter $m$ is set to $-\frac{1}{2}$ in the solution, Eq.(14) [9].


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# Figure Captions

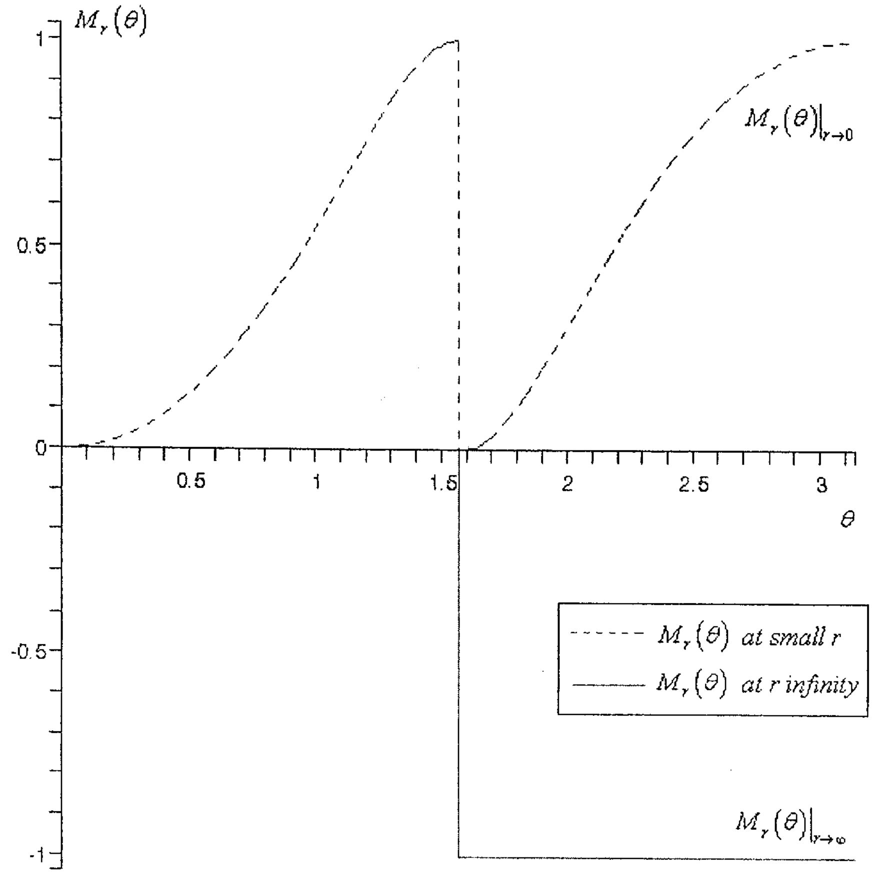


Figure 1: A plot of $M_r(\theta)$ at small $r$ close to zero and $M_r(\theta)$ at $r$ infinity when $m = 1$, versus $\theta$.

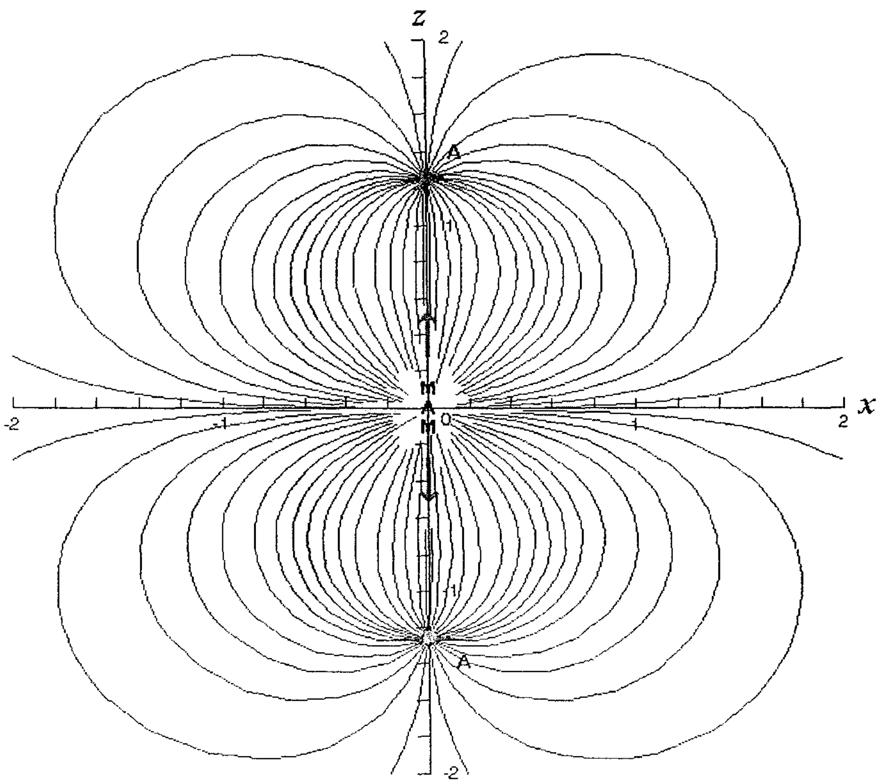


Figure 2: A plot of the magnetic field lines when $m = 1$ along a vertical plane through the z-axis. At large $r$, all the field lines are concentrated radially inwards along the equatorial plane. The two antimonopoles are located along the z-axis at $z = \pm 1.2599$.

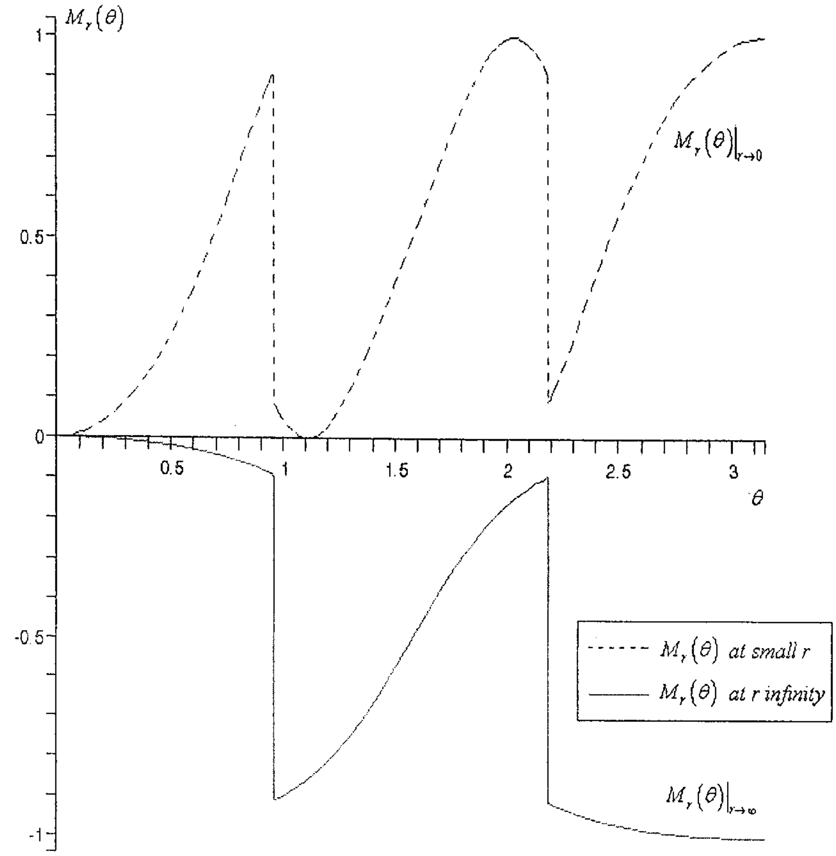


Figure 3: A plot of $M_r(\theta)$ at $r$ close to zero and $M_r(\theta)$ at $r$ infinity, when $m = 2$, versus $\theta$.

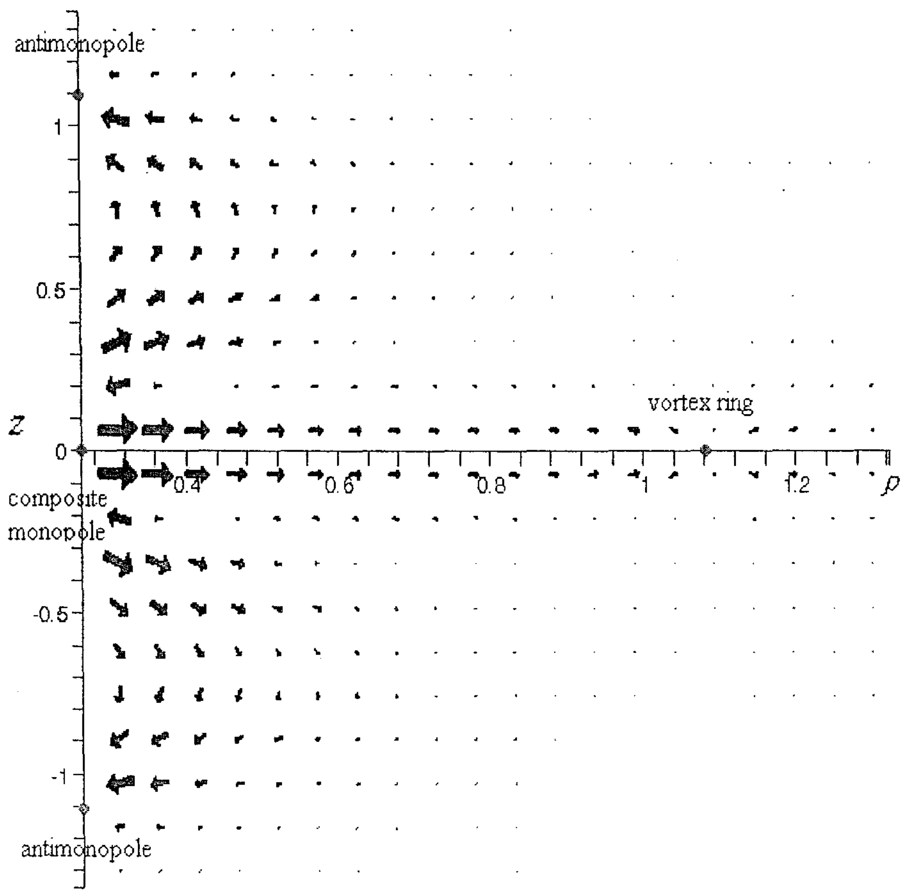


Figure 4: The Abelian magnetic field of the $m = 2$ solution at finite distances showing the presence of the two dipole pairs along the z-axis and the vortex ring at $z = 0,\ \rho = 1.0845$.

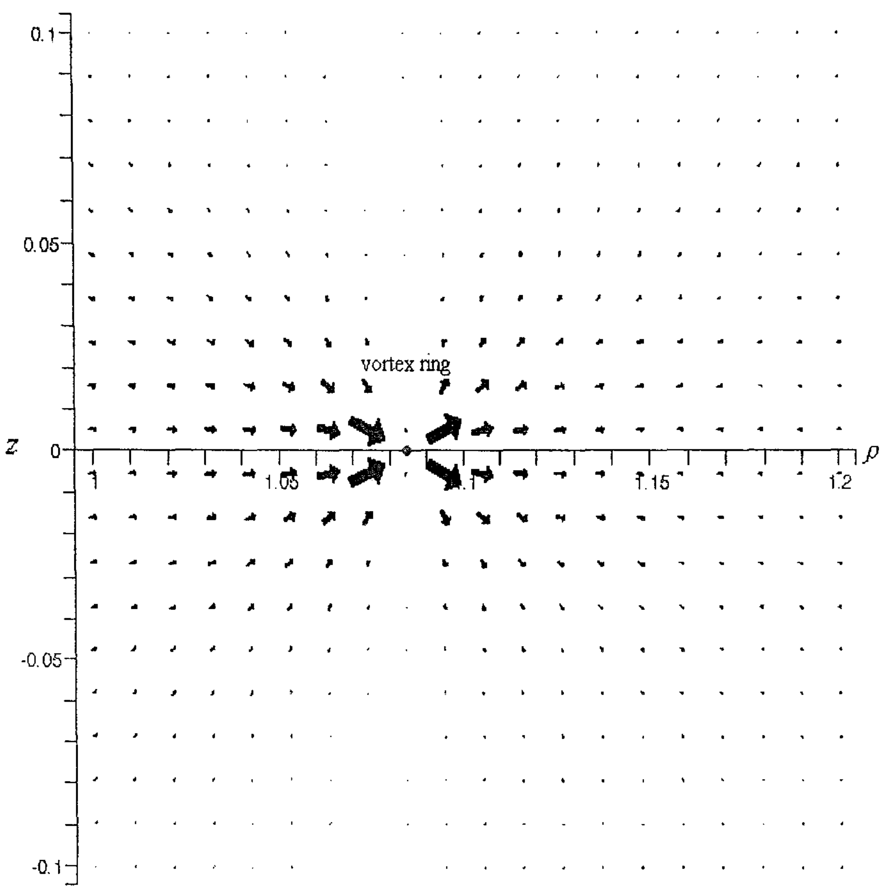


Figure 5: The Abelian magnetic field of the $m = 2$ solution at distances close to $z = 0$ and $\rho = 1.0845$ showing the presence of the vortex ring.

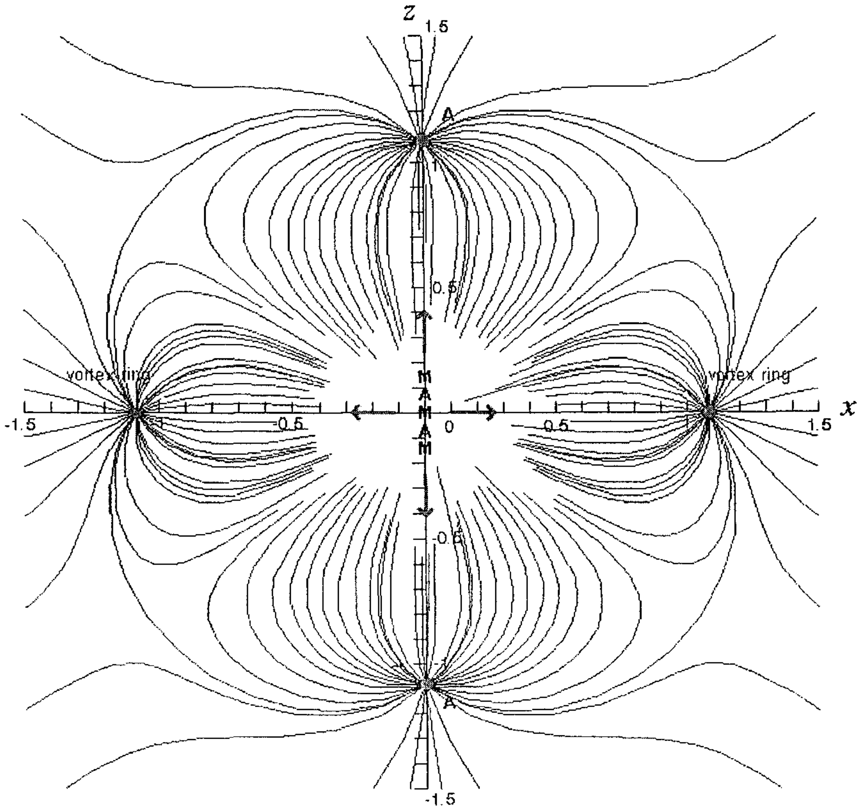


Figure 6: A plot of the magnetic field lines when $m = 2$ along a vertical plane through the z-axis. The vortex ring is situated horizontally at equal distances from the origin as the two antimonopoles at $z = \pm 1.0845$.

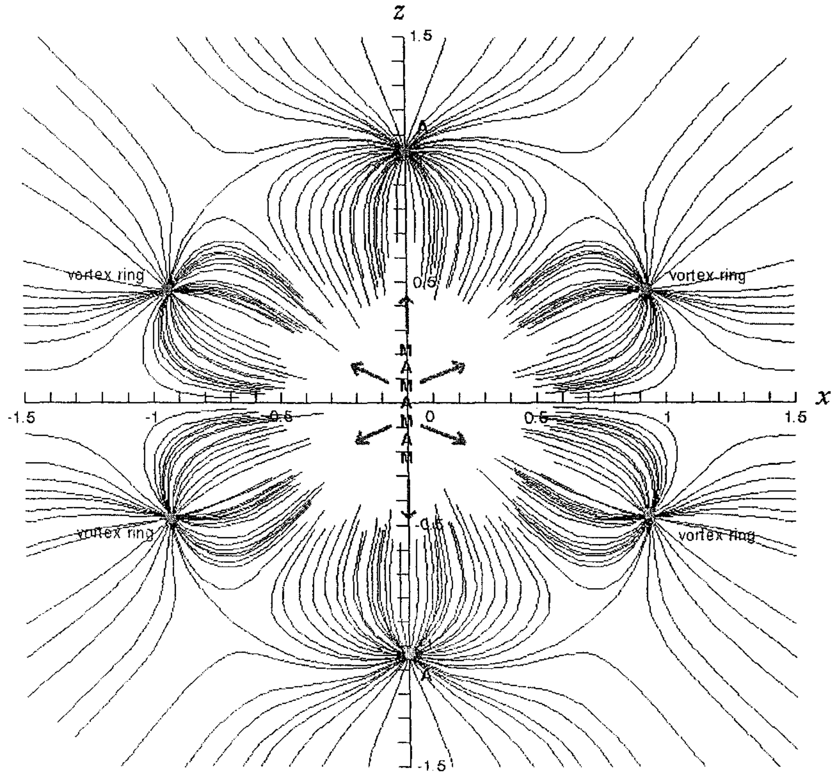


Figure 8: A plot of the magnetic field lines when $m = 3$ along a vertical plane through the z-axis. The two vortex rings are situated horizontally at equal distances from the origin as the two antimonopoles at $z = \pm 1.0420$.